

POLYNOMIALS IN OPERATOR SPACE THEORY : MATRIX ORDERING AND ALGEBRAIC ASPECTS

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ABSTRACT. We extend the λ -theory of operator spaces (based on the tensor norms obtained from homogeneous polynomials) given in [4], which generalizes the notion of the projective, Haagerup and Schur tensor norm for operator spaces, to matrix ordered spaces and Banach $*$ -algebras. Given matrix (ordered) regular operator space and (unital) operator systems, we introduce cones related to λ for the algebraic operator space tensor product to be matrix (ordered) regular and (unital) operator system, respectively. The ideal structure in λ -tensor product of C^* -algebras is also discussed.

1. INTRODUCTION

C^* -algebras are rich objects in the sense that they come along with matrix norms that are not only uniquely related to algebraic structure, but are also known to have matricial cone structures being closely related to those norm. Although, operator spaces and their tensor products are primarily defined in terms of appropriate matrix norms, over the years it has been observed that some operator space tensor products of C^* -algebras still possess few algebraic properties that can be characterized in terms of the individual algebras ([1, 14]). Regarding ordering, although operator spaces may possess some order structure unrelated to the matrix norms, it was Schreiner [18] who defined matrix regular operator spaces to be the spaces where there is a relationship between norm and order. In matrix regular operator spaces, there are enough positive elements so that each element can be written as a linear combination of positive elements. Recently introduced tensor product theory for (unital) operator systems category ([13]) shows that this matrix order-matrix norm relation is successfully carried over.

Defant and Wiesner in [4] (see also [19]) have given a λ -theory which generalizes the definitions of the projective, Haagerup and Schur tensor norm for operator spaces. It is thus natural to ask for appropriate matrix ordering and algebraic structure that is compatible with this generalized λ -theory. In [9] and [17], the projective and Schur operator space tensor product of matrix ordered operator spaces are shown to be matrix ordered respectively. Further, in [8] Han successfully introduced cones at each matrix level of the tensor product of operator spaces that are closely related to projective and injective operator space tensor norms. In this way he could construct two extremal tensor products of matrix regular operator space.

After giving all prerequisites in Section 2, we introduce conditions (O1) – (O3) in Section 3 that helps us to generalize Han's ([8]) operator space tensor product

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matrix regularity results to λ -theory of operator spaces.

In Section 4, we show that the cones defined in Section 3 also preserve the operator system structure.

Finally in the last section, we show that the techniques to study ideal structure of operator space tensor product of C^* -algebras can be extended to λ -theory also.

2. PRELIMINARIES

2.1. The λ -theory [4][19]. Let V_1, V_2, \dots, V_m be operator spaces and let ϕ be a m -linear mapping on $V_1 \times V_2 \times \dots \times V_m$. Given a sequence of matrix products $\lambda = (\lambda_n)$, i.e., for each k some m -linear mapping

$$\lambda_k \in L(M_k \times \dots \times M_k, M_{\tau(k)}),$$

where $\tau(k) \in \mathbb{N}$ is a natural number only depending on k , tensorizing λ_k with ϕ leads to the m -linear mapping

$$\begin{aligned} \phi_{\lambda_k} &:= \lambda_k \otimes \phi : M_k(V_1) \otimes \dots \otimes M_k(V_m) \rightarrow M_{\tau(k)}(\otimes_{i=1}^m V_i), \\ (\alpha_1 \otimes v_1, \dots, \alpha_m \otimes v_m) &\mapsto \lambda_k(\alpha_1, \dots, \alpha_m) \otimes \phi(v_1, v_2, \dots, v_m). \end{aligned}$$

Now, since m -fold tensor product on $V_1 \times V_2 \times \dots \times V_m$ is an m -linear map, the natural map obtained as above by tensorizing with λ_k is represented by \otimes_{λ_j} .

In [4, 19], a tensor norm λ was defined as:

$$(1) \quad \|u\|_{\lambda, k} = \inf \{ \|\alpha\| \|v_1\| \|v_2\| \dots \|v_m\| \|\beta\| \}$$

for any element $u \in M_k(\otimes_{i=1}^m V_i)$, where the infimum is taken over arbitrary decompositions $u = \alpha \otimes_{\lambda_j} (v_1, v_2, \dots, v_m) \beta$, $\alpha \in M_{k, \tau(j)}$, $\beta \in M_{\tau(j), k}$, $v_t \in M_j(V_t)$. Three technical conditions (E1) – (E3) were isolated on the family $\lambda = (\lambda_n)_{n \in \mathbb{N}}$ to assure that the $\|\cdot\|_{\lambda, k}$, generates an operator space structure on $V_1 \otimes V_2 \otimes \dots \otimes V_m$ [4, Proposition 4.1]

(E1) For all $k \in \mathbb{N}$ there exist $p \in \mathbb{N}$ and matrices $S \in M_{k, \tau(p)}$, $T \in M_{\tau(p), k}$, $a_1, \dots, a_k \in M_p$ such that for all $j_1, \dots, j_m \in \{1, \dots, k\}$:

$$S \lambda_p(a_{j_1}, \dots, a_{j_m}) T = \begin{cases} \varepsilon_j^{[k]} & \text{if } j_1 = j_2 = \dots = j_m = j, \\ 0 & \text{otherwise} \end{cases}$$

(E2) For all $r, s \in \mathbb{N}$ there exist matrices $P \in M_{\tau(r) + \tau(s), \tau(r+s)}$, with $\|P\| \leq 1$ such that for all $(i_k, j_k) \in \{1, \dots, r\}^2 \cup \{r+1, \dots, r+s\}^2$ with $1 \leq k \leq m$:

$$\begin{aligned} &P \lambda_{r+s}(\varepsilon_{i_1, j_1}^{[r+s]}, \dots, \varepsilon_{i_m, j_m}^{[r+s]}) P^* \\ &= \text{diag} \left(\lambda_r(\varepsilon_{i_1, j_1}^{[r]}, \dots, \varepsilon_{i_m, j_m}^{[r]}), \lambda_s(\varepsilon_{i_1-r, j_1-r}^{[s]}, \dots, \varepsilon_{i_m-r, j_m-r}^{[s]}) \right). \end{aligned}$$

(E3) $\lambda_1(1, 1, \dots, 1) = 1$ and $\sup_{k \in \mathbb{N}} \|\lambda_k\| < \infty$.

If in addition λ satisfies:

(N1) $\tau(1) = 1$ and (N2) $\|\lambda_j\| = 1$ for all $j \in \mathbb{N}$
then $\otimes_{\lambda} \mathbb{C} = \mathbb{C}$ completely isometric [19, Proposition 4.13].

For $j \in \{1, \dots, m\}$, if λ further satisfies conditions:

(W1) For all $\gamma \in M_p$ there exists matrices $P \in M_{p, \tau(p)}$, $Q \in M_{\tau(p), p}$ with $\|P\|, \|Q\| \leq 1$ such that

$$\gamma = P \lambda_p(I_p, \dots, I_p, \underbrace{\gamma}_{\text{j-th position}}, I_p, \dots, I_p) Q.$$

(W2) For all $\alpha_1, \dots, \alpha_m \in M_p$, $\beta_1, \dots, \beta_m \in M_q$ there exist matrices $S \in M_{\tau(p)\tau(q), \tau(pq)}$, $T \in M_{\tau(pq), \tau(p)\tau(q)}$ with $\|S\|, \|T\| \leq 1$ such that

$$\lambda_p(\alpha_1, \dots, \alpha_m) \otimes \lambda_q(\beta_1, \dots, \beta_m) = S \lambda_{pq}(\alpha_1 \otimes \beta_1, \dots, \alpha_m \otimes \beta_m) T.$$

then the mapping

$$\begin{aligned} \Phi^{(j)} : (V_1 \otimes \dots \otimes M_p(V_j) \otimes \dots \otimes V_m, \|\cdot\|_\lambda) &\rightarrow M_p(\otimes_\lambda V_i) \\ v_1 \otimes \dots \otimes (\alpha \otimes v_j) \otimes \dots \otimes v_m &\mapsto \alpha \otimes (v_1 \otimes \dots \otimes v_m) \end{aligned}$$

is completely contractive [19, Proposition 12.2].

If λ satisfies (N1) – (N2), (E1) – (E3) and (W1) – (W2) then $\otimes^\lambda V_i$, the completion of $\otimes_\lambda V_i$ with respect to $\|\cdot\|_\lambda$ norm, is an operator space tensor product denoted by λ -operator space tensor product in the sense of [3].

The Kronecker product, the matrix product and the mixed product fulfill all the above conditions.

In this paper whenever we talk about λ -tensor norm it is assumed that λ satisfies all the prescribed conditions.

The following results regarding dual will be used:

Proposition 2.1. [19, Proposition 5.3] *If λ satisfies conditions (E1) – (E3), then for $\|u\|_{\lambda^*, k} = \sup\{\|\otimes_{\lambda_p}(f_1, f_2, \dots, f_m)_k(u)\|_{M_k(M_{\tau(p)})} : f_i \in CB(V_i, M_p)_1, i = 1, 2, \dots, m, p \in \mathbb{N}\}$.*

We keep the notations from [19, 4] unchanged: $\varepsilon_{i,j} := \varepsilon_{i,j}^{[k,l]} \in M_{k,l}$ denotes the matrix which is 1 in the (i, j) -th coordinate and zero elsewhere. Moreover, $\varepsilon_{i,j}^{[k]} := \varepsilon_{i,j}^{[k,k]}$, $\varepsilon_i := \varepsilon_{i,i}$, and $\varepsilon_i^{[k]} := \varepsilon_{i,i}^{[k,k]}$. For all $k \in \mathbb{N}$ define $I_k := \sum_{i=1}^k \varepsilon_i^{[k]}$ as well as $e_j := e_j^{[k]} := \varepsilon_{j,1}^{[k,1]}$. Also, note that $\varepsilon_{i,j}^{[k,l]} = 0$ if $(i, j) \notin \{1, \dots, k\} \times \{1, \dots, l\}$.

2.2. Matrix regular operator space and operator systems. An operator space V is called a matrix ordered operator space if

- (1) $(V, \{M_n(V)^+\}_{n=1}^\infty)$ is a matrix ordered vector space (i.e. for each $n \in \mathbb{N}$, $M_n(V)$ is a $*$ -ordered vector space with cone $M_n(V)^+$ and $A \in M_{n,m}$ implies $A^* M_n(V)^+ A \subseteq M_m(V)^+$.)
- (2) the $*$ -operation is an isometry on $M_n(V)$, and
- (3) the cones $M_n(V)^+$ are closed.

A matrix ordered operator space V is matrix regular [18, Definition 3.1.9] if for each $n \in \mathbb{N}$ and for all $v \in M_n(V)_{sa}$, the following conditions hold :

- (1) $u \in M_n(V)^+$ and $-u \leq v \leq u$ implies that $\|v\|_n \leq \|u\|_n$.
- (2) $\|v\|_n \leq 1$ there exists $u \in M_n(V)^+$ such that $\|u\|_n \leq 1$ and $-u \leq v \leq u$.

We'll use the following necessary and sufficient for matrix regularity [18, Theorem 3.4]:

A matrix ordered operator space V is matrix regular if and only if the following

condition holds: for all $x \in M_n(V)$, $\|x\|_n < 1$ if and only if there exist $a, d \in M_n(V)^+$, $\|a\|_n < 1$ and $\|d\|_n < 1$, such that $\begin{bmatrix} a & x \\ x^* & d \end{bmatrix} \in M_{2n}(V)^+$. The positive cone of a matrix regular operator space is always proper.

We are adopting here the methodology of [8] where the norms on matrix regular operator spaces were not assumed be complete.

For a matrix ordered operator space V and its dual space V^* , the positive cone on $M_n(V^*)$ for each $n \in \mathbb{N}$ is defined by $M_n(V^*)^+ = CB(V, M_n) \cap CP(V, M_n)$. The operator space dual V^* with this positive cone is a matrix ordered operator space [18, Corollary 3.2].

An (abstract) operator system ([13, Definition 2.2]) is a triple $(V, \{\mathcal{C}_n\}_{n=1}^\infty, e)$, where V is a complex $*$ -vector space, $\{\mathcal{C}_n\}_{n=1}^\infty$ is a matrix ordering on V , and $e \in V_h$ is a matrix order unit, that is, (i) if for all $v \in M_n(V)_h$ there exists a real number $r > 0$ such that $re_n > v$ and an Archimedean matrix order unit it it also satisfies (ii) $re_n + v \in M_n(V)^+$; where $e_n = \begin{pmatrix} e & & \\ & \ddots & \\ & & e \end{pmatrix}$ for all $r > 0$ implies $v \in M_n(V)^+$.

3. MATRIX REGULARITY AND λ -THEORY

In this section, we provide three additional conditions on $\lambda = (\lambda_n)_{n \in \mathbb{N}}$ to introduce an order structure to λ -theory that preserves matrix regularity. In fact, using our conditions (O1)-(O3) defined below, we prove that the results of [8] hold true in a more general setting introduced by [4, 19].

For a sequence $\lambda = (\lambda_n)_{n \in \mathbb{N}}$ of m -linear mappings $\lambda_k \in L({}^m M_k; M_{\tau(k)})$ consider the following three properties:

(O1) For each $r \in \mathbb{N}$,

$$\lambda_r(\varepsilon_{i_1, j_1}^{[r]}, \varepsilon_{i_2, j_2}^{[r]}, \dots, \varepsilon_{i_m, j_m}^{[r]}) = \lambda_r(\varepsilon_{j_1, i_1}^{[r]}, \varepsilon_{j_2, i_2}^{[r]}, \dots, \varepsilon_{j_m, i_m}^{[r]})^* \in M_{\tau(r)},$$

for all $(i_k, j_k) \in \{1, \dots, r\} \times \{1, \dots, r\}$, and $k = 1, 2, \dots, m$.

(O2) For each $r \in \mathbb{N}$ and $(i_k, j_k) \in R \cup S$, where $R := \{1, \dots, r\} \times \{r+1, r+2, \dots, 2r\}$ and $S := \{r+1, r+2, \dots, 2r\} \times \{1, \dots, r\}$

$$\begin{aligned} & P \lambda_{2r}(\varepsilon_{i_1, j_1}^{[2r]}, \dots, \varepsilon_{i_1, j_1}^{[2r]}) P^* \\ &= \text{adiag}(\lambda_r(\varepsilon_{i_1, j_1-r}^{[r]}, \dots, \varepsilon_{i_1, j_1-r}^{[r]}), \lambda_r(\varepsilon_{i_1-r, j_1}^{[r]}, \dots, \varepsilon_{i_m-r, j_m}^{[r]})), \end{aligned}$$

the anti-diagonal matrix where all the entries are zero except those on the diagonal going from the upper right corner to the lower left corner with the permutation matrix $P \in M_{2\tau(r), \tau(2r)}$ with $\|P\| \leq 1$ is same as obtained in (E2) for $r = s$ under the set conditions on r there.

(O3) For each $r \in \mathbb{N}$, the map,

$$\bigotimes_{\lambda_r}^{p_1 \dots p_m} = \bigotimes_{\lambda_r}^{p_1 \dots p_m} \otimes \lambda_r : M_r(M_{p_1}) \otimes M_r(M_{p_2}) \otimes \dots \otimes M_r(M_{p_m}) \rightarrow M_{\tau(r)}(M_{p_1 p_2 \dots p_m}),$$

obtained by tensorizing λ_r with the Kronecker product on Matrix algebras M_{p_1}, \dots, M_{p_m} ,

$$\begin{aligned} \bigotimes_{\lambda_r}^{p_1 \dots p_m} : M_{p_1} \times M_{p_2} \times \dots \times M_{p_m} &\rightarrow M_{p_1 p_2 \dots p_m} \\ (\alpha_1, \alpha_2, \dots, \alpha_m) &\mapsto \alpha_1 \otimes \alpha_2 \otimes \dots \otimes \alpha_m, \end{aligned}$$

is positive for all $p_i \in \mathbb{N}$ ($i = 1, 2, \dots, m$). Thus

$$\bigotimes_{\lambda_r}^{p_1 \dots p_m} (\alpha_1 \otimes \beta_1, \dots, \alpha_m \otimes \beta_m) = \lambda_r(\beta_1 \otimes \dots \otimes \beta_m) \otimes \alpha_1 \otimes \dots \otimes \alpha_m \in M_{\tau(r)p_1 p_2 \dots p_m}^+,$$

whenever $\alpha_i \otimes \beta_i \in (M_{p_i} \otimes M_r)^+$, $p_i \in \mathbb{N}$ ($i = 1, 2, \dots, m$).

Recall from [18, Proposition 4.1] (see also [19, proposition 4.2]), given a sequence $\lambda = (\lambda_n)$ of m -linear maps and operator spaces V_1, V_2, \dots, V_m , any element $u \in M_k(V_1 \otimes \dots \otimes V_m)$ has a representation $u = \alpha \otimes_{\lambda_r} (v_1, v_2, \dots, v_m)\beta$ where $\alpha \in M_{n, \tau(r)}$, $\beta \in M_{\tau(r), n}$, $v_i \in M_r(V_i)$, $r \in \mathbb{N}$. Now we analyze the above conditions in view of their application to matrix ordered spaces:

Lemma 3.1. *Let $\lambda = (\lambda_n)$ be sequence of m -linear maps and V_1, \dots, V_m be matrix ordered operator spaces. Then for any $\alpha \otimes_{\lambda_r} (v^{(1)}, v^{(2)}, \dots, v^{(m)})\beta \in M_n(\otimes_{\lambda} V_i)$; $\alpha \in M_{n, \tau(r)}$, $\beta \in M_{\tau(r), n}$, $v^{(i)} \in M_r(V_i)$, $i = 1, \dots, m$, $r \in \mathbb{N}$,*

(i) *If λ satisfies (O1), $*$ -map defined as*

$$(\alpha \otimes_{\lambda_r} (v^{(1)}, v^{(2)}, \dots, v^{(m)})\beta)^* = \beta^* \otimes_{\lambda_r} ((v^{(1)})^*, (v^{(2)})^*, \dots, (v^{(m)})^*)\alpha^*,$$

is a well defined involution.

(ii) *If λ satisfies (O2), then for $u^{(i)}, \tilde{u}^{(i)} \in M_r(V_i)$, $i = 1, \dots, m$, we have*

$$\begin{aligned} &\begin{pmatrix} \otimes_{\lambda_r} (u^{(1)}, u^{(2)}, \dots, u^{(m)}) & \otimes_{\lambda_r} (v^{(1)}, v^{(2)}, \dots, v^{(m)}) \\ \otimes_{\lambda_r} ((v^{(1)})^*, (v^{(2)})^*, \dots, (v^{(m)})^*) & \otimes_{\lambda_r} (\tilde{u}^{(1)}, \tilde{u}^{(2)}, \dots, \tilde{u}^{(m)}) \end{pmatrix} \\ &= P \otimes_{\lambda_{2r}} \left(\begin{pmatrix} u^{(1)} & v^{(1)} \\ (v^{(1)})^* & \tilde{u}^{(1)} \end{pmatrix}, \dots, \begin{pmatrix} u^{(m)} & v^{(m)} \\ (v^{(m)})^* & \tilde{u}^{(m)} \end{pmatrix} \right) P^*. \end{aligned}$$

(iii) *If λ and μ are such that $(\mu_p)_{\lambda_r}$ is a positive map ($p, r \in \mathbb{N}$), then for $v^{(i)} \in M_r(V_i)^+$ and completely positive maps $\phi^{(i)} : V_i \rightarrow M_{p_i}$, $i = 1, \dots, m$, we have*

$$(\otimes_{\mu_p} (\phi^{(1)}, \dots, \phi^{(m)}))_n (\alpha \otimes_{\lambda_r} (v^{(1)}, v^{(2)}, \dots, v^{(m)})\alpha^*) \in M_{np_1 \dots p_m}^+.$$

In particular, if λ satisfies (O3), we have

$$((\phi^{(1)} \otimes \dots \otimes \phi^{(m)}))_n (\alpha \otimes_{\lambda_r} (v^{(1)}, v^{(2)}, \dots, v^{(m)})\alpha^*) \in M_{np_1 \dots p_m}^+.$$

Proof. (i) One can easily verify that the $*$ -operation is conjugate linear and involutive.

(ii) Since

$$\begin{aligned} &\begin{pmatrix} \otimes_{\lambda_r} (u^{(1)}, u^{(2)}, \dots, u^{(m)}) & \otimes_{\lambda_r} (v^{(1)}, v^{(2)}, \dots, v^{(m)}) \\ \otimes_{\lambda_r} ((v^{(1)})^*, (v^{(2)})^*, \dots, (v^{(m)})^*) & \otimes_{\lambda_r} (\tilde{u}^{(1)}, \tilde{u}^{(2)}, \dots, \tilde{u}^{(m)}) \end{pmatrix} \\ &= \text{diag}(\otimes_{\lambda_r} (u^{(1)}, u^{(2)}, \dots, u^{(m)}), \otimes_{\lambda_r} (\tilde{u}^{(1)}, \tilde{u}^{(2)}, \dots, \tilde{u}^{(m)})) + \\ &\quad \text{adiag}(\otimes_{\lambda_r} (v^{(1)}, v^{(2)}, \dots, v^{(m)}), \otimes_{\lambda_r} ((v^{(1)})^*, (v^{(2)})^*, \dots, (v^{(m)})^*)) \end{aligned}$$

Set $R_1 := \{1, 2, \dots, r\}^2$, $R_2 := \{1, 2, \dots, r\} \times \{r+1, r+2, \dots, 2r\}$,
 $R_3 := \{r+1, r+2, \dots, 2r\} \times \{1, 2, \dots, r\}$ and $R_4 := \{r+1, r+2, \dots, 2r\}^2$.
Let $u^{(t)} := \sum_{(k_t, l_t) \in R_1} \varepsilon_{k_t, l_t}^{[r]} \otimes u_{k_t, l_t}^{(t)}$, $v^{(t)} := \sum_{(k_t, l_t) \in R_2} \varepsilon_{k_t, l_t-r}^{[r]} \otimes v_{k_t, l_t}^{(t)}$,
 $(v^{(t)})^* = \sum_{(k_t, l_t) \in R_3} \varepsilon_{k_t-r, l_t}^{[r]} \otimes (v_{k_t, l_t}^{(t)})^*$ and $\tilde{u}^{(t)} := \sum_{(k_t, l_t) \in R_4} \varepsilon_{k_t-r, l_t}^{[r]} \otimes \tilde{u}_{k_t, l_t}^{(t)}$.
Define $x^{(t)} := \text{diag}(u^{(t)}, \tilde{u}^{(t)})$ and $y^{(t)} := \text{adiag}(v^{(t)}, (v^{(t)})^*)$, so that

$$x_{k,l}^{(t)} = \begin{cases} u_{k,l}^{(t)} & \text{if } (k, l) \in R_1, \\ \tilde{u}_{k,l}^{(t)} & \text{if } (k, l) \in R_4 \end{cases} \text{ and } y_{k,l}^{(t)} = \begin{cases} v_{k,l}^{(t)} & \text{if } (k, l) \in R_2, \\ (v^{(t)})_{k,l}^* & \text{if } (k, l) \in R_3 \end{cases}.$$

Then,

$$\begin{aligned} & \text{adiag}(\otimes_{\lambda_r} (v^{(1)}, v^{(2)}, \dots, v^{(m)}), 0) \\ &= \text{adiag}(\sum_{(k_m, l_m) \in R_2} \dots \sum_{(k_1, l_1) \in R_2} \lambda_r(\varepsilon_{k_1, l_1-r}^{[r]}, \dots, \varepsilon_{k_m, l_m-r}^{[r]}) \otimes v_{k_1, l_1}^{(1)} \otimes \dots \otimes v_{k_m, l_m}^{(m)}, 0) \\ &= \sum_{(k_m, l_m) \in R_2} \dots \sum_{(k_1, l_1) \in R_2} \text{adiag}(\lambda_r(\varepsilon_{k_1, l_1-r}^{[r]}, \dots, \varepsilon_{k_m, l_m-r}^{[r]}), 0) \otimes y_{k_1, l_1}^{(1)} \otimes \dots \otimes y_{k_m, l_m}^{(m)} \\ &\stackrel{(\mathbf{O2})}{=} \sum_{(k_m, l_m) \in R_2} \dots \sum_{(k_1, l_1) \in R_2} P \lambda_{2r}(\varepsilon_{k_1, l_1}^{[2r]}, \dots, \varepsilon_{k_m, l_m}^{[2r]}) P^* \otimes y_{k_1, l_1}^{(1)} \otimes \dots \otimes y_{k_m, l_m}^{(m)} \end{aligned}$$

Also,

$$\begin{aligned} & \text{adiag}(0, \otimes_{\lambda_r} ((v^{(1)})^*, (v^{(2)})^*, \dots, (v^{(m)})^*)) \\ &= \text{adiag}(0, \sum_{(k_m, l_m) \in R_3} \dots \sum_{(k_1, l_1) \in R_3} \lambda_r(\varepsilon_{k_1-r, l_1}^{[r]}, \dots, \varepsilon_{k_m-r, l_m}^{[r]}) \otimes (v_{k_1, l_1}^{(1)})^* \otimes \dots \otimes (v_{k_m, l_m}^{(m)})^*) \\ &= \sum_{(k_m, l_m) \in R_3} \dots \sum_{(k_1, l_1) \in R_3} \text{adiag}(0, \lambda_r(\varepsilon_{k_1-r, l_1}^{[r]}, \dots, \varepsilon_{k_m-r, l_m}^{[r]}) \otimes y_{k_1, l_1}^{(1)} \otimes \dots \otimes y_{k_m, l_m}^{(m)}) \\ &\stackrel{(\mathbf{O2})}{=} \sum_{(k_m, l_m) \in R_3} \dots \sum_{(k_1, l_1) \in R_3} P \lambda_{2r}(\varepsilon_{k_1, l_1}^{[2r]}, \dots, \varepsilon_{k_m, l_m}^{[2r]}) P^* \otimes y_{k_1, l_1}^{(1)} \otimes \dots \otimes y_{k_m, l_m}^{(m)} \end{aligned}$$

Thus

$$\begin{aligned} & \text{adiag}(\otimes_{\lambda_r} (v^{(1)}, v^{(2)}, \dots, v^{(m)}), \otimes_{\lambda_r} ((v^{(1)})^*, (v^{(2)})^*, \dots, (v^{(m)})^*)) \\ &= \sum_{(k_m, l_m) \in R_2 \cup R_3} \dots \sum_{(k_1, l_1) \in R_2 \cup R_3} P \lambda_{2r}(\varepsilon_{k_1, l_1}^{[2r]}, \dots, \varepsilon_{k_m, l_m}^{[2r]}) P^* \otimes y_{k_1, l_1}^{(1)} \otimes \dots \otimes y_{k_m, l_m}^{(m)} \end{aligned}$$

Similarly, using (E2),

$$\begin{aligned} & \text{diag}(\otimes_{\lambda_r} (u^{(1)}, u^{(2)}, \dots, u^{(m)}), \otimes_{\lambda_r} (\tilde{u}^{(1)}, \tilde{u}^{(2)}, \dots, \tilde{u}^{(m)})) \\ &= \sum_{(k_m, l_m) \in R_1 \cup R_4} \dots \sum_{(k_1, l_1) \in R_1 \cup R_4} P \lambda_{2r}(\varepsilon_{k_1, l_1}^{[2r]}, \dots, \varepsilon_{k_m, l_m}^{[2r]}) P^* \otimes x_{k_1, l_1}^{(1)} \otimes \dots \otimes x_{k_m, l_m}^{(m)} \end{aligned}$$

Therefore,

$$\begin{aligned}
& \begin{pmatrix} \otimes_{\lambda_r}(u^{(1)}, u^{(2)}, \dots, u^{(m)}) & \otimes_{\lambda_r}(v^{(1)}, v^{(2)}, \dots, v^{(m)}) \\ \otimes_{\lambda_r}((v^{(1)})^*, (v^{(2)})^*, \dots, (v^{(m)})^*) & \otimes_{\lambda_r}(\tilde{u}^{(1)}, \tilde{u}^{(2)}, \dots, \tilde{u}^{(m)}) \end{pmatrix} \\
&= P \left(\sum_{(k_m, l_m) \in \cup_{i=1}^4 R_i} \dots \sum_{(k_1, l_1) \in \cup_{i=1}^4 R_i} \otimes_{\lambda_{2r}}(\varepsilon_{k_1, l_1}^{[2r]} \otimes x_{k_1, l_1}^{(1)} + \varepsilon_{k_1, l_1}^{[2r]} \otimes y_{k_1, l_1}^{(1)}, \dots \right. \\
&\quad \left. \dots, \varepsilon_{k_m, l_m}^{[2r]} \otimes x_{k_m, l_m}^{(m)} + \varepsilon_{k_m, l_m}^{[2r]} \otimes y_{k_m, l_m}^{(m)}) \right) P^* \\
&= P \otimes_{\lambda_{2r}} \left(\begin{pmatrix} u^{(1)} & v^{(1)} \\ (v^{(1)})^* & \tilde{u}^{(1)} \end{pmatrix}, \dots, \begin{pmatrix} u^{(m)} & v^{(m)} \\ (v^{(m)})^* & \tilde{u}^{(m)} \end{pmatrix} \right) P^*.
\end{aligned}$$

(iii) Note that, if $v^{(t)} := \sum_{(k_t, l_t) \in R} \varepsilon_{k_t, l_t}^{[r]} \otimes v_{k_t, l_t}^{(t)}$; $R := \{1, \dots, r\}^2$, $t = 1, 2, \dots, m$,

$$\begin{aligned}
& (\otimes_{\mu_p}(\phi^{(1)}, \dots, \phi^{(m)})_{\tau_{\mu}(p)}(\otimes_{\lambda_r}(v^{(1)}, v^{(2)}, \dots, v^{(m)}))) \\
&= \sum_{k_m, l_m} \dots \sum_{k_1, l_1} (\otimes_{\mu_p}(\phi^{(1)}, \dots, \phi^{(m)})_{\tau_{\mu}(p)}(\lambda_r(\varepsilon_{k_1, l_1}, \dots, \varepsilon_{k_m, l_m}) \otimes v_{k_1, l_1}^{(1)} \otimes \dots \otimes v_{k_m, l_m}^{(m)})) \\
&= \sum_{k_m, l_m} \dots \sum_{k_1, l_1} \lambda_r(\varepsilon_{k_1, l_1}, \dots, \varepsilon_{k_m, l_m}) \otimes (\otimes_{\mu_p}(\phi^{(1)}, \dots, \phi^{(m)})(v_{k_1, l_1}^{(1)} \otimes \dots \otimes v_{k_m, l_m}^{(m)})) \\
&= \sum_{k_m, l_m} \dots \sum_{k_1, l_1} \lambda_r(\varepsilon_{k_1, l_1}, \dots, \varepsilon_{k_m, l_m}) \otimes \mu_p(\phi^{(1)}(v_{k_1, l_1}^{(1)}) \otimes \dots \otimes \phi^{(m)}(v_{k_m, l_m}^{(m)})) \\
&= \sum_{k_m, l_m} \dots \sum_{k_1, l_1} (\mu_p)_{\lambda_r}(\varepsilon_{k_1, l_1} \otimes \phi^{(1)}(v_{k_1, l_1}^{(1)}), \dots, \varepsilon_{k_m, l_m} \otimes \phi^{(m)}(v_{k_m, l_m}^{(m)})) \\
&= \sum_{k_m, l_m} \dots \sum_{k_1, l_1} (\mu_p)_{\lambda_r}(\phi_r^{(1)}(\varepsilon_{k_1, l_1} \otimes v_{k_1, l_1}^{(1)}), \dots, \phi_r^{(m)}(\varepsilon_{k_m, l_m} \otimes v_{k_m, l_m}^{(m)})) \\
&= (\mu_p)_{\lambda_r}(\phi_r^{(1)}(v^{(1)}), \dots, \phi_r^{(m)}(v^{(m)})) \\
&\in M_{\tau(r)p_1 p_2 \dots p_m}^+,
\end{aligned}$$

so that

$$\begin{aligned}
& (\otimes_{\mu_p}(\phi^{(1)}, \dots, \phi^{(m)})_n(\alpha \otimes_{\lambda_r}(v^{(1)}, \dots, v^{(m)})\alpha^*)) \\
&= \alpha((\otimes_{\mu(p)}(\phi^{(1)}, \dots, \phi^{(m)}))_{\tau_{\mu}(p)}(\otimes_{\lambda_j}(v_1, v_2, \dots, v_m)))\alpha^* \in M_{nk_1 k_2 \dots k_m}^+.
\end{aligned}$$

If λ satisfies (O3), $\mu_p = \overset{p_1 \dots p_m}{\otimes}$ gives the desired result. \square

Verification of Properties (O1)-(O3):

- *Kronecker product:* It satisfies all of the condition (O1)-(O3):
Property (O1) reduces to

$$\varepsilon_{i_1, j_1}^{[r]} \otimes \varepsilon_{i_2, j_2}^{[r]} \otimes \dots \otimes \varepsilon_{i_m, j_m}^{[r]} = (\varepsilon_{j_1, i_1}^{[r]} \otimes \varepsilon_{j_2, i_2}^{[r]} \otimes \dots \otimes \varepsilon_{j_m, i_m}^{[r]})^*,$$

which is true.

To check for the condition (O3), recall that Kronecker product of two positive matrices is positive, but Kronecker product does not commute, infact for any square matrices A and B , there exist a permutaion matrix S such

that $B \otimes A = S(A \otimes B)S^*$. Therefore, for some suitable permutation matrix S we have:

$$\begin{aligned} \bigotimes_{\otimes_r}^{p_1 \dots p_m} (\alpha_1 \otimes \beta_1, \dots, \alpha_m \otimes \beta_m) &= (\beta_1 \otimes \dots \otimes \beta_m) \otimes \alpha_1 \otimes \dots \otimes \alpha_m \\ &= S((\alpha_1 \otimes \beta_1) \otimes \dots \otimes (\alpha_m \otimes \beta_m))S^* \\ &\in M_{r^m p_1 p_2 \dots p_m}^+, \end{aligned}$$

whenever $\alpha_i \otimes \beta_i \in M_r(M_{p_i})^+$, $p_i \in \mathbb{N}$ ($i = 1, 2, \dots, m$).

In order to verify (O2), we use the same notations as in Proof of [4, 4.2], let $\Delta : M_1 \rightarrow M_{1,m}$, $x \mapsto (x, x, \dots, x)$ and set

$$P_1 := \sum_{r \in \{1, 2, \dots, r\}^m} \varepsilon_{p,p}^{[\Delta r, \Delta 2r]} \quad \text{and} \quad P_2 := \sum_{r \in \{r+1, r+2, \dots, 2r\}^m} \varepsilon_{p-\Delta r, p}^{[\Delta r, \Delta 2r]},$$

$$\text{and let } P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

$$\begin{aligned} &P \otimes_{2r} (\varepsilon_{i_1, j_1}^{[2r]}, \dots, \varepsilon_{i_m, j_m}^{[2r]}) P^* \\ &= P \varepsilon_{(i_1, \dots, i_m), (j_1, \dots, j_m)}^{[\Delta 2r, \Delta 2r]} P^* \\ &= \begin{pmatrix} \varepsilon_{(i_1, \dots, i_m), (j_1, j_2, \dots, j_m)}^{[\Delta r, \Delta r]} & \varepsilon_{(i_1, \dots, i_m), (j_1, j_2, \dots, j_m) - \Delta r}^{[\Delta r, \Delta r]} \\ \varepsilon_{(i_1, \dots, i_m) - \Delta r, (j_1, j_2, \dots, j_m)}^{[\Delta r, \Delta r]} & \varepsilon_{(i_1, \dots, i_m) - \Delta r, (j_1, j_2, \dots, j_m) - \Delta r}^{[\Delta r, \Delta r]} \end{pmatrix} \\ &= \begin{cases} \text{adiag}(\varepsilon_{(i_1, \dots, i_m), (j_1, j_2, \dots, j_m) - \Delta r}^{[\Delta r, \Delta r]}, 0) & \text{if } (i_k, j_k) \in \{1, \dots, r\} \times \{r+1, \dots, 2r\}, \\ \text{adiag}(0, \varepsilon_{(i_1, \dots, i_m) - \Delta r, (j_1, j_2, \dots, j_m)}^{[\Delta r, \Delta r]}) & \text{if } (i_k, j_k) \in \{r+1, \dots, 2r\} \times \{1, \dots, r\} \\ 0 & \text{else} \end{cases} \end{aligned}$$

- *Schur product*: It satisfies all the condition (O1)-(O3). Property (O1) reduces to

$$\varepsilon_{i_1, j_1}^{[r]} \odot \varepsilon_{i_2, j_2}^{[r]} \odot \dots \odot \varepsilon_{i_m, j_m}^{[r]} = (\varepsilon_{j_1, i_1}^{[r]} \odot \varepsilon_{j_2, i_2}^{[r]} \odot \dots \odot \varepsilon_{j_m, i_m}^{[r]})^*,$$

which is true.

To check for the condition (O3), recall that Schur product of two positive matrices is positive, for any square matrices A, B, C and D of order n , $(A \odot B) \otimes (C \odot D) = (A \otimes C) \odot (B \otimes D)$ (see [19, Proposition 10.5]) and there exist a matrix $\mathcal{E} \in M_{n,k}$, $A', B' \in M_k$ such that $(A \otimes B) = \mathcal{E}(A' \odot B')\mathcal{E}^*$. Therefore we have,

$$\begin{aligned} \bigotimes_{\odot_r}^{p_1 \dots p_m} (\alpha_1 \otimes \beta_1, \dots, \alpha_m \otimes \beta_m) &= (\beta_1 \odot \dots \odot \beta_m) \otimes \alpha_1 \otimes \dots \otimes \alpha_m \\ &\in M_{r p_1 p_2 \dots p_m}^+, \end{aligned}$$

whenever $\alpha_i \otimes \beta_i \in M_r(M_{p_i})^+$, $p_i \in \mathbb{N}$ ($i = 1, 2, \dots, m$).

Again as for (E2), for $r \in \mathbb{N}$ and $(i_q, j_q) \in [\{1, \dots, r\} \times \{r+1, \dots, 2r\}] \cup$

$[\{r+1, \dots, 2r\} \times \{1, \dots, r\}]$, let $P := I_{2r}$, we have

$$\begin{aligned} & P \odot_{2r} (\varepsilon_{i_1, j_1}^{[2r]}, \dots, \varepsilon_{i_m, j_m}^{[2r]}) P^* \\ &= \begin{pmatrix} 0 & \varepsilon_{i_1, j_1-r}^{[r]} \\ \varepsilon_{i_1-r, j_1}^{[r]} & 0 \end{pmatrix} \odot \dots \odot \begin{pmatrix} 0 & \varepsilon_{i_m, j_m-r}^{[r]} \\ \varepsilon_{i_m-r, j_m}^{[r]} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \varepsilon_{i_1, j_1-r}^{[r]} \odot \dots \odot \varepsilon_{i_m, j_m-r}^{[r]} \\ \varepsilon_{i_1-r, j_1}^{[r]} \odot \dots \odot \varepsilon_{i_m-r, j_m}^{[r]} & 0 \end{pmatrix} \end{aligned}$$

implying that (O2) holds.

- *Matrix Product*: One can easily see that this product does not satisfies (O1), (O2) and (O3).
- *Mixed Product*: One can mix above listed products to construct a new one, for example [19, Chapter 9], $\lambda = (\lambda_k)_k$ with

$$\lambda_k : M_k \times M_k \times M_k \times M_k \rightarrow M_k^2,$$

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \mapsto (\alpha_1 \bullet \alpha_2) \otimes (\alpha_3 \bullet \alpha_4).$$

Clearly it does not satisfy any of the (O1) – (O3) as \bullet does not.

One can similarly talk of $\lambda = (\lambda_k)_k$ with

$$\lambda_k : M_k \times M_k \times M_k \times M_k \rightarrow M_k^2,$$

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \mapsto (\alpha_1 \odot \alpha_2) \otimes (\alpha_3 \odot \alpha_4),$$

which apparently satisfies all the conditions (O1) – (O3).

The self-adjoint elements in $M_n(\otimes_\lambda V_i)$ have a special representation:

Proposition 3.2. *Let V_i , $i = 1, 2, \dots, m$, be matrix ordered operator spaces and let $\lambda = (\lambda_n)_{n \in \mathbb{N}}$ be a sequence of m -multilinear mappings satisfying (O1) and (O2). If $u \in M_n(\otimes_\lambda V_i)_{sa}$ then u has a representation*

$$u = \alpha \otimes_{\lambda_j} (x_1, x_2, \dots, x_m) \alpha^*,$$

where $\alpha \in M_{n, \tau(j)}$, $x_t \in M_j(V_t)_{sa}$, $t = 1, 2, \dots, m$. Moreover,

$$\begin{aligned} \|u\|_{\lambda, n} &= \inf \{ \|\alpha\|^2 \|x_1\| \|x_2\| \dots \|x_m\| : u = \alpha \otimes_{\lambda_j} (x_1, x_2, \dots, x_m) \alpha^*, \\ &\quad \alpha \in M_{n, \tau(j)}, x_t \in M_j(V_t)_{sa}, j \in \mathbb{N} \}. \end{aligned}$$

Proof. Suppose $u \in M_n(\otimes_\lambda V_i)_{sa}$. Given $\epsilon > 0$, there exist $\alpha \in M_{n, \tau(j)}$, $\beta \in M_{\tau(j), n}$, $x_t \in M_j(V_t)$ such that

$$\begin{aligned} u &= \alpha \otimes_{\lambda_j} (x_1, x_2, \dots, x_m) \beta \quad \text{with} \\ \|u\|_{\lambda, n} &\leq \|\alpha\| \|x_1\| \|x_2\| \dots \|x_m\| \|\beta\| \leq \|u\|_{\lambda, n} + \epsilon. \end{aligned}$$

As u is self adjoint, by Lemma 3.1(ii) for any $\mu > 0$, we have

$$\begin{aligned}
u &= \frac{1}{2}(u + u^*) \\
&= \frac{1}{2}(\mu\alpha \otimes_{\lambda_j} (x_1, x_2, \dots, x_m)\mu^{-1}\beta + \mu^{-1}\beta^* \otimes_{\lambda_j} (x_1^*, x_2^*, \dots, x_m^*)\mu\alpha^*) \\
&= \left(\begin{array}{cc} \frac{\mu^{-1}\beta^*}{\sqrt{2}} & \frac{\mu\alpha}{\sqrt{2}} \end{array} \right) \left(\begin{array}{cc} 0 & \otimes_{\lambda_j}(x_1^*, x_2^*, \dots, x_m^*) \\ \otimes_{\lambda_j}(x_1, x_2, \dots, x_m) & 0 \end{array} \right) \left(\begin{array}{c} \frac{\mu^{-1}\beta}{\sqrt{2}} \\ \frac{\mu\alpha^*}{\sqrt{2}} \end{array} \right) \\
&= \alpha P(\otimes_{\lambda_{2j}}(v^1, v^2, \dots, v^m))P^*\alpha^*,
\end{aligned}$$

where $v^t = \begin{pmatrix} 0 & x_t^* \\ x_t & 0 \end{pmatrix}$, and $\alpha = \begin{pmatrix} \frac{\mu^{-1}\beta^*}{\sqrt{2}} & \frac{\mu\alpha}{\sqrt{2}} \end{pmatrix}$, $R = \{1, 2, \dots, r\} \times \{r+1, \dots, 2r\}$ and $S = \{r+1, r+2, \dots, 2r\} \times \{1, \dots, r\}$.

Thus, $\|u\|_{\lambda, n} \leq \frac{1}{2}(\mu^2\|\beta\|^2 + \mu^{-2}\|\alpha\|^2)\|v^1\|\|v^2\|\dots\|v^m\|$, where for each t , v^t is a self adjoint element.

Now, by using the fact that $\min_{\mu>0} \frac{1}{2}(\mu^2\|\beta\|^2 + \mu^{-2}\|\alpha\|^2) = \|\beta\|\|\alpha\|$, given $\delta > 0$ choose $\mu_0 > 0$ such that $\|\beta\|\|\alpha\| + \delta > \frac{1}{2}(\mu_0^2\|\beta\|^2 + \mu_0^{-2}\|\alpha\|^2)$. We then have $\|u\|_{\lambda, n} \leq \|\tilde{\alpha}\|^2\|v^1\|\|v^2\|\dots\|v^m\| \leq \|\beta\|\|\alpha\|\|x_1\|\|x_2\|\dots\|x_m\|$. Thus, we get the desired norm condition. \square

We are now in a position to define an appropriate cone structure :

Definition 3.3. Let V_1, \dots, V_m be matrix ordered operator spaces and $\lambda = (\lambda_n)_{n \in \mathbb{N}}$ be sequence satisfying (O1) – (O2):

- (a) Define $\mathcal{C}_n = \{\alpha \otimes_{\lambda_j} (v_1, v_2, \dots, v_m)\alpha^* : v_t \in M_j(V_t)^+, \alpha \in M_{n, \tau(j)}, j \in \mathbb{N}, t = 1, 2, \dots, m\} \subset M_n(\otimes_{\lambda} V_i)_{sa}$.
- (b) $M_n(\otimes_{\lambda} V_i)^+ := \mathcal{C}_n^{-\|\cdot\|_{\lambda, n}}$.

Proposition 3.4. For matrix ordered operator spaces V_i , $(\otimes_{i=1}^m V_i, \{\|\cdot\|_{\lambda, n}\}_{n=1}^\infty, M_n(\otimes_{\lambda} V_i)^+)$ is a matrix ordered operator space.

Proof. From Proposition 3.2, the involution is an isometry on $(M_n(\otimes_{\lambda} V_i))_{sa}$ and hence $M_n(\otimes_{\lambda} V_i)^+$ is a cone provided \mathcal{C}_n is.

Thus let $u_1 = \alpha_1 \otimes_{\lambda_k} (v_1, v_2, \dots, v_m)\alpha_1^* \in \mathcal{C}_{n_1}$, $u_2 = \alpha_2 \otimes_{\lambda_l} (w_1, w_2, \dots, w_m)\alpha_2^* \in \mathcal{C}_n$. Then, using Lemma 3.1(ii), we have

$$\begin{aligned}
u_1 + u_2 &= \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \begin{pmatrix} \otimes_{\lambda_k}(v_1, v_2, \dots, v_m) & 0 \\ 0 & \otimes_{\lambda_l}(w_1, w_2, \dots, w_m) \end{pmatrix} \begin{pmatrix} \alpha_1^* \\ \alpha_2^* \end{pmatrix} \\
&= \alpha P(\otimes_{\lambda_{k+l}}(x_1, x_2, \dots, x_m))P^*\alpha^* \in \mathcal{C}_n,
\end{aligned}$$

where $x_t = \text{diag}(v_t, w_t) \in M_{k+l}(V)^+$ and $\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix}$, and hence the family $\{\mathcal{C}_n\}$ is closed under addition.

Now, for $t \geq 0$

$$\begin{aligned}
tu_1 &= t\alpha_1 \otimes_{\lambda_k} (v_1, v_2, \dots, v_m)\alpha_1^* \\
&= t^{1/2}\alpha_1 \otimes_{\lambda_k} (v_1, v_2, \dots, v_m)t^{1/2}\alpha_1^* \in \mathcal{C}_n.
\end{aligned}$$

Also, for $\gamma \in M_{m, n}$ and $\alpha \otimes_{\lambda_k} (v_1, v_2, \dots, v_m)\alpha^* \in \mathcal{C}_m$,

$$\begin{aligned}
&\gamma^* \alpha \otimes_{\lambda_k} (v_1, v_2, \dots, v_m)\alpha^* \gamma \\
&= \gamma^* \alpha \otimes_{\lambda_k} (v_1, v_2, \dots, v_m)(\gamma^* \alpha)^* \in \mathcal{C}_n.
\end{aligned}$$

\square

Remark 3.5. Note that $\begin{pmatrix} u & z \\ z^* & u' \end{pmatrix} \in \mathcal{C}_{2n}$, implies that

$$u = \begin{pmatrix} 1_n & 0 \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} u & z \\ z^* & u' \end{pmatrix} \begin{pmatrix} 1_n \\ 0 \end{pmatrix} \in \mathcal{C}_n \quad \text{and} \quad u' = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} u & z \\ z^* & u' \end{pmatrix} \begin{pmatrix} 0 \\ 1_n \end{pmatrix} \in \mathcal{C}_n,$$

so that using Proposition 3.2

$$\begin{aligned} \|u\|_{\lambda,n} &= \inf \{ \|\alpha\|^2 \|x_1\| \|x_2\| \cdots \|x_m\| : u = \alpha \otimes_{\lambda_j} (x_1, x_2, \dots, x_m) \alpha^*, \\ &\quad \alpha \in M_{n,\tau(j)}, x_t \in M_j(V_t)^+, j \in \mathbb{N} \} \end{aligned}$$

and

$$\begin{aligned} \|u'\|_{\lambda,n} &= \inf \{ \|\alpha'\|^2 \|x'_1\| \|x'_2\| \cdots \|x'_m\| : u = \alpha' \otimes_{\lambda_r} (x'_1, x'_2, \dots, x'_m) \alpha'^*, \\ &\quad \alpha' \in M_{n,\tau(r)}, x'_t \in M_r(V_t)^+, j \in \mathbb{N} \}. \end{aligned}$$

Motivated by this and Han's [8, Definition 3.2], we relate a suitable norm to the cone \mathcal{C}_n defined above that behaves well with matrix regular operator spaces.

Definition 3.6. Let V_1, \dots, V_m be matrix ordered operator spaces and $\lambda = (\lambda_n)_{n \in \mathbb{N}}$ be sequence satisfying (O1) – (O3). Then for z in $M_n(\otimes_{\lambda} V_i)$, define

$$\|z\|_{\Lambda,n} := \inf \{ \max \{ \|u\|_{\lambda,n}, \|u'\|_{\lambda,n} \} : \begin{pmatrix} u & z \\ z^* & u' \end{pmatrix} \in \mathcal{C}_{2n} \}.$$

Note that if $u \in \mathcal{C}_n$, then

$$\begin{pmatrix} u & u \\ u & u \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} u \begin{pmatrix} 1 & 1 \end{pmatrix} \in \mathcal{C}_{2n},$$

therefore $\|\cdot\|_{\Lambda,n} \leq \|\cdot\|_{\lambda,n}$ on \mathcal{C}_n . Also the set $\{ \max \{ \|u\|_{\lambda,n}, \|u'\|_{\lambda,n} \} : \begin{pmatrix} u & z \\ z^* & u' \end{pmatrix} \in \mathcal{C}_{2n} \}$ is non empty from Proposition 3.7(i) and (ii) proved below.

Proposition 3.7. If V_1, V_2, \dots, V_m are matrix regular operator spaces and $\lambda = (\lambda_n)_{n \in \mathbb{N}}$ be sequence satisfying (O1) – (O3).

(i) \mathcal{C}_n is a proper cone in $M_n(\otimes_{i=1}^m V_i)$ for all $n \in \mathbb{N}$.

(ii) For $z \in M_n(\otimes_{i=1}^m V_i)$, there exist elements u_1, u_2 in \mathcal{C}_n such that $\begin{pmatrix} u_1 & z \\ z^* & u_2 \end{pmatrix} \in \mathcal{C}_{2n}$.

(iii) $\|\cdot\|_{\Lambda,n}$ is a norm on $M_n(\otimes_{i=1}^m V_i)$.

Proof. (i) Let $z \in \mathcal{C}_n \cap -\mathcal{C}_n$.

Then $z = \alpha \otimes_{\lambda_j} (v^1, v^2, \dots, v^m) \alpha^*$ with $v^t \in M_j(V_t)^+, \alpha \in M_{n,\tau(j)}$, and by Lemma 3.1(iii), for continuous c.p. maps $\phi^t : V_t \rightarrow M_{k_t}, t = 1, \dots, m$,

$$(\otimes_{t=1}^m \phi^t)_n(z) \in M_{nk_1 k_2 \dots k_m}^+ \cap -M_{nk_1 k_2 \dots k_m}^+ = \{0\}.$$

Now each V_t being matrix regular its dual V_t^* is also matrix regular ([18, Corollary 6.7]), hence each completely bounded linear map from V_t into a matrix algebra is actually a linear combination of some completely positive maps.

Thus, even for c.b. maps $f^{(t)} : V_t \rightarrow M_{k_t}, t = 1, \dots, m$;

$$(\otimes_{t=1}^m f^{(t)})_n(z) \in M_{nk_1 k_2 \dots k_m}^+ \cap -M_{nk_1 k_2 \dots k_m}^+ = \{0\},$$

which further implies the operator space injective tensor norm is given by

$$\|z\|_{M_n(\otimes V_i)} = 0, \text{ giving } z = 0.$$

- (ii) Using [4, Proposition 4.1], any $z \in M_n(\otimes_{i=1}^m V_i)$ can be written as:
 $z = \alpha \otimes_{\lambda_j} (v_1, v_2, \dots, v_m) \beta^*$ for $v_t \in M_j(V_t)$, $\alpha, \beta \in M_{n, \tau(j)}$, $t = 1, 2, \dots, m$.
 As each V_i is a matrix regular operator space, there exist $v_t^1, v_t^2 \in M_j(V_t)^+$ such that

$$x_t = \begin{pmatrix} v_t^1 & v_t \\ v_t^* & v_t^2 \end{pmatrix} \in M_{2j}(V_t)^+, t = 1, 2, \dots, m.$$

Using Lemma 3.1(ii), we have

$$\begin{aligned} & \begin{pmatrix} \alpha \otimes_{\lambda_j} (v_1^1, v_2^1, \dots, v_m^1) \alpha^* & \alpha \otimes_{\lambda_j} (v_1, v_2, \dots, v_m) \beta^* \\ \beta \otimes_{\lambda_j} (v_1^*, v_2^*, \dots, v_m^*) \alpha^* & \beta \otimes_{\lambda_j} (v_1^2, v_2^2, \dots, v_m^2) \beta^* \end{pmatrix} \\ &= \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} \otimes_{\lambda_j} (v_1^1, v_2^1, \dots, v_m^1) & \otimes_{\lambda_j} (v_1, v_2, \dots, v_m) \\ \otimes_{\lambda_j} (v_1^*, v_2^*, \dots, v_m^*) & \otimes_{\lambda_j} (v_1^2, v_2^2, \dots, v_m^2) \end{pmatrix} \begin{pmatrix} \alpha^* & 0 \\ 0 & \beta^* \end{pmatrix} \\ &= \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \otimes_{\lambda_{2j}} \left(\begin{pmatrix} v_1^1 & v_1 \\ v_1^* & v_1^2 \end{pmatrix}, \begin{pmatrix} v_2^1 & v_2 \\ v_2^* & v_2^2 \end{pmatrix}, \dots, \begin{pmatrix} v_m^1 & v_m \\ v_m^* & v_m^2 \end{pmatrix} \right) \begin{pmatrix} \alpha^* & 0 \\ 0 & \beta^* \end{pmatrix} \in \mathcal{C}_{2n}. \end{aligned}$$

- (iii) $\|\cdot\|_{\Lambda, n}$ clearly satisfies homogeneity.

Let $z_1, z_2 \in M_n(\otimes_{i=1}^m V_i)$ be any elements. Then by (ii) above there exist $u_i, u'_i \in \mathcal{C}_n$ such that

$$\begin{pmatrix} u_i & z_i \\ z_i^* & u'_i \end{pmatrix} \in \mathcal{C}_{2n} \quad \text{and} \quad \|u_i\|_{\lambda, n}, \|u'_i\|_{\lambda, n} < \|z\|_{\Lambda, n} + \epsilon.$$

From

$$\begin{pmatrix} u_1 + u_2 & z_1 + z_2 \\ z_1^* + z_2^* & u'_1 + u'_2 \end{pmatrix} \in \mathcal{C}_{2n}$$

it follows that

$$\begin{aligned} \|z_1 + z_2\|_{\Lambda, n} &\leq \max\{\|u_1 + u_2\|_{\lambda, n}, \|u'_1 + u'_2\|_{\lambda, n}\} \\ &\leq \max\{\|u_1\|_{\lambda, n} + \|u_2\|_{\lambda, n}, \|u'_1\|_{\lambda, n} + \|u'_2\|_{\lambda, n}\} \\ &< \|z_1\|_{\Lambda, n} + \|z_2\|_{\Lambda, n} + 2\epsilon \end{aligned}$$

Let $\|z\|_{\Lambda, n} = 0$. Then given $\epsilon > 0$, there exist $u, u' \in \mathcal{C}_n$ such that

$$\begin{pmatrix} u & z \\ z^* & u' \end{pmatrix} \in \mathcal{C}_{2n} \quad \text{and} \quad \|u\|_{\lambda, n}, \|u'\|_{\lambda, n} < \epsilon.$$

Again by Lemma 3.1(iii), for c.c.p. maps $f^{(t)} : V_t \rightarrow M_{k_t}$, $t = 1, \dots, m$;

$$\begin{aligned} & \begin{pmatrix} (\otimes_{t=1}^m f^{(t)})_n(u) & (\otimes_{t=1}^m f^{(t)})_n(z) \\ (\otimes_{t=1}^m f^{(t)})_n(z^*) & (\otimes_{t=1}^m f^{(t)})_n(u') \end{pmatrix} \\ &= (\otimes_{t=1}^m f^{(t)})_{2n} \left(\begin{pmatrix} u & z \\ z^* & u' \end{pmatrix} \right) \in M_{2nk_1 k_2 \dots k_m}^+. \end{aligned}$$

It follows that

$$\begin{aligned} \|(\otimes_{t=1}^m f^{(t)})_n(z)\| &\leq \max\{\|(\otimes_{t=1}^m f^{(t)})_n(u)\|, \|(\otimes_{t=1}^m f^{(t)})_n(u')\|\} \\ &\leq \max\{\|u\|_{\lambda, n}, \|u'\|_{\lambda, n}\} < \epsilon. \end{aligned}$$

Thus $(\otimes_{t=1}^m f^{(t)})_n(z) = 0$ and using matrix regularity as in case (i), $\|z\|_{M_n(\otimes V_i)} = 0$, which implies $z = 0$. □

Since the positive cones of matrix ordered operator spaces are closed, we consider $M_n(\otimes_{\Lambda} V_i)^+ := \mathcal{C}_n^{\|\cdot\|_{\Lambda,n}}$. From the definition of $\|\cdot\|_{\Lambda,n}$, since $\begin{pmatrix} u & z \\ z^* & u' \end{pmatrix} \in \mathcal{C}_{2n}$ if and only if $\begin{pmatrix} u' & z^* \\ z & u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u & z \\ z^* & u' \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathcal{C}_{2n}$, we have $\|z^*\|_{\Lambda,n} = \|z\|_{\Lambda,n}$. Thus the involution is an isometry on $(M_n(\otimes_{i=1}^m V_i), \|\cdot\|_{\Lambda,n})$, and hence $M_n(\otimes_{\Lambda} V_i)^+$ is a proper cone.

Theorem 3.8. *For matrix regular operator spaces V_i and $\lambda = (\lambda_n)_{n \in \mathbb{N}}$ satisfying (O1)-(O3), $(\otimes_{i=1}^m V_i, \{\|\cdot\|_{\Lambda,n}\}_{n=1}^{\infty}, M_n(\otimes_{\Lambda} V_i)^+)$ is a matrix regular operator space with λ -subcross matrix norm.*

Proof. We first prove that $M_n(\otimes_{i=1}^m V_i)$ is an operator space with the family $\{\|\cdot\|_{\Lambda,n}\}_{n=1}^{\infty}$ of matrix norms.

Given $z_1 \in M_{n_1}(\otimes_{i=1}^m V_i)$, $z_2 \in M_{n_2}(\otimes_{i=1}^m V_i)$ and $\epsilon > 0$, choose $u_i, u'_i \in \mathcal{C}_n$ such that $\begin{pmatrix} u_i & z_i \\ z_i^* & u'_i \end{pmatrix} \in \mathcal{C}_{2n_i}$ and $\|u_i\|_{\lambda,n_i}, \|u'_i\|_{\lambda,n_i} < \|z_i\|_{\Lambda,n_i} + \epsilon$, $i = 1, 2$.

By definition, there exist representations

$$u_i = \alpha_i \otimes_{\lambda_{j_i}} (v_1^i, v_2^i, \dots, v_m^i) \alpha_i^* \quad \alpha_i \in M_{n_i, \tau(j_i)}, v_t^i \in M_{j_i}(V_t)^+.$$

Since $\|\cdot\|_{\lambda,n}$ is an operator space norm, using Ruan's first condition (M1) [5] for operator space, we have

$$\|u_1 \oplus u_2\|_{\lambda, 2n} \leq \max\{\|u_1\|_{\lambda,n}, \|u_2\|_{\lambda,n}\}.$$

As

$$\begin{aligned} & \begin{pmatrix} u_1 & 0 & z_1 & 0 \\ 0 & u_2 & 0 & z_2 \\ z_1^* & 0 & u_1' & 0 \\ 0 & z_2^* & 0 & u_2' \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 & z_1 & 0 & 0 \\ z_1^* & u_1' & 0 & 0 \\ 0 & 0 & u_2 & z_2 \\ 0 & 0 & z_2^* & u_2' \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{C}_{2(n_1+n_2)}. \end{aligned}$$

Thus

$$\begin{aligned} \left\| \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \right\|_{\Lambda, 2n} &\leq \max\left\{ \left\| \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \right\|_{\lambda, 2n}, \left\| \begin{pmatrix} u_1' & 0 \\ 0 & u_2' \end{pmatrix} \right\|_{\lambda, 2n} \right\} \\ &\leq \max\{\|u_1\|_{\lambda,n}, \|u_2\|_{\lambda,n}, \|u_3\|_{\lambda,n}, \|u_4\|_{\lambda,n}\} \\ &< \max\{\|z_1\|_{\Lambda,n}, \|z_2\|_{\Lambda,n}\} + \epsilon. \end{aligned}$$

Let $z \in M_n(\otimes_{i=1}^m V_i)$ and $\alpha, \beta \in M_{m,n}$. Then there exist $u, u' \in \mathcal{C}_n$ such that $\begin{pmatrix} u & z \\ z^* & u' \end{pmatrix} \in \mathcal{C}_{2n}$ and $\|u\|_{\lambda,n}, \|u'\|_{\lambda,n} < \|z\|_{\Lambda,n} + \epsilon$. Assuming $\|\alpha\| = \|\beta\|$ by homogeneity, since

$$\begin{pmatrix} \alpha u \alpha^* & \alpha z \beta \\ \beta^* z^* \alpha^* & \beta^* u' \beta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta^* \end{pmatrix} \begin{pmatrix} u & z \\ z^* & u' \end{pmatrix} \begin{pmatrix} \alpha^* & 0 \\ 0 & \beta \end{pmatrix} \in \mathcal{C}_{2n},$$

we have

$$\begin{aligned}\|\alpha z \beta\|_{\Lambda, n} &\leq \max\{\|\alpha u \alpha^*\|_{\lambda, n}, \|\beta^* u' \beta\|\} \\ &\leq \max\{\|\alpha\|^2 \|u\|_{\lambda, n}, \|\beta\|^2 \|u'\|_{\lambda, n}\} \\ &< \|\alpha\| \|\beta\| (\|z\|_{\Lambda, n} + \epsilon).\end{aligned}$$

Hence, $(\otimes_{i=1}^m V_i, \{\|\cdot\|_{\Lambda, n}\}_{n=1}^\infty)$ is an operator space.

Let $v_i \in M_j(V_i)$ with $\|v_i\| < 1$ for $i = 1, \dots, m$. Then there exist $u_i, u'_i \in M_j(V_i)_{\|\cdot\| \leq 1}^+$ such that

$$\begin{pmatrix} u_i & v_i \\ v_i^* & u'_i \end{pmatrix} \in M_{2j}(V_i)^+ \quad i = 1, 2, \dots, m.$$

Now,

$$\begin{aligned}&\begin{pmatrix} \otimes_{\lambda_j}(u_1, \dots, u_m) & \otimes_{\lambda_j}(v_1, \dots, v_m) \\ \otimes_{\lambda_j}(v_1, \dots, v_m)^* & \otimes_{\lambda_j}(u'_1, \dots, u'_m) \end{pmatrix} \\ &= P \otimes_{\lambda_{2j}} \left(\begin{pmatrix} u_1 & v_1 \\ v_1^* & u'_1 \end{pmatrix}, \dots, \begin{pmatrix} u_m & v_m \\ v_m^* & u'_m \end{pmatrix} \right) P^* \in \mathcal{C}_{2\tau(j)},\end{aligned}$$

it follows that

$$\begin{aligned}\|\otimes_{\lambda_j}(v_1, \dots, v_m)\|_{\Lambda, 2j} &\leq \max\{\|\otimes_{\lambda_j}(u_1, \dots, u_m)\|_{\lambda, j}, \|\otimes_{\lambda_j}(u'_1, \dots, u'_m)\|_{\lambda, j}\} \\ &\leq \max\{\|u_1\| \dots \|u_m\|, \|u'_1\| \dots \|u'_m\|\} \\ &< 1\end{aligned}$$

Therefore, the family of matrix norms $\{\|\cdot\|_{\Lambda, n}\}_{n=1}^\infty$ is λ -subcross.

If $u \in \mathcal{C}_n$, then

$$\begin{pmatrix} u & u \\ u & u \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} u \begin{pmatrix} 1 & 1 \end{pmatrix} \in \mathcal{C}_{2n},$$

therefore $\|u\|_{\Lambda, n} \leq \|u\|_{\lambda, n}$. Let $\|z\|_{\Lambda, n} < 1$. Then there exist $u, u' \in \mathcal{C}_n$ such that

$$\begin{pmatrix} u & z \\ z^* & u' \end{pmatrix} \in \mathcal{C}_{2n} \quad \text{and} \quad \|u\|_{\lambda, n}, \|u'\|_{\lambda, n} < 1.$$

Since

$$\|u\|_{\Lambda, n} \leq \|u\|_{\lambda, n} < 1 \quad \text{and} \quad \|u'\|_{\Lambda, n} \leq \|u'\|_{\lambda, n} < 1,$$

matrix regularity follows. \square

4. λ -OPERATOR SYSTEM TENSOR PRODUCT

We now prove that the cones \mathcal{C}_n associated with λ under the conditions (O1)-(O3) also preserve the operator system structure defined in [13]. The techniques are again same as that for the max operator system tensor product defined in [13].

Theorem 4.1. *Let $(\mathcal{S}, \{M_n(\mathcal{S})^+\}_{n=1}^\infty, 1_{\mathcal{S}})$ and $(\mathcal{T}, \{M_n(\mathcal{T})^+\}_{n=1}^\infty, 1_{\mathcal{T}})$ be operator systems. The family $\{\mathcal{C}_n\}_{n=1}^\infty$ associated with a sequence $\lambda = (\lambda_n)_{n \in \mathbb{N}}$ satisfying (O1)-(O3), is a matrix ordering on $\mathcal{S} \otimes \mathcal{T}$ with order unit $1_{\mathcal{S}} \otimes 1_{\mathcal{T}}$.*

Proof. From Proposition 3.7 we know that $\{\mathcal{C}_n\}_{n=1}^\infty$ is a family of proper compatible cones on $M_n(\otimes_{\lambda} \mathcal{S}_i)$. We only need to check that $1 \otimes 1$ is a matrix order unit. Let $\alpha \otimes_{\lambda_j}(s, t) \alpha^* \in (\mathcal{S}_1 \otimes \mathcal{S}_2)_h$ with $s \in (M_j(\mathcal{S}))_h$, $t \in (M_j(\mathcal{T}))_h$ and $\alpha \in M_{1, \tau(j)}$. Then $1_{\mathcal{S}}$ and $1_{\mathcal{T}}$ being Archimedean order unit for \mathcal{S} and \mathcal{T} respectively, we can find K large enough such that

$$K(1_{\mathcal{S}})_j + s \in M_j(\mathcal{S})^+ \quad \text{and} \quad K(1_{\mathcal{S}})_j - s \in M_j(\mathcal{S})^+,$$

$$K(1_{\mathcal{T}})_j + t \in M_j(\mathcal{T})^+ \quad \text{and} \quad K(1_{\mathcal{T}})_j - t \in M_j(\mathcal{T})^+.$$

So that

$\otimes_{\lambda_j}(K(1_{\mathcal{S}})_j + s, K(1_{\mathcal{T}})_j + t), \otimes_{\lambda_j}(K(1_{\mathcal{S}})_j - s, K(1_{\mathcal{T}})_j - t) \in \mathcal{C}_n$. Now, consider

$$\begin{aligned} \mathcal{C}_n \ni & \alpha \otimes_{\lambda_j}(K(1_{\mathcal{S}})_j + s, K(1_{\mathcal{T}})_j + t) \alpha^* + \alpha \otimes_{\lambda_j}(K(1_{\mathcal{S}})_j - s, K(1_{\mathcal{T}})_j - t) \alpha^* \\ &= \alpha \left(\otimes_{\lambda_j}(K(1_{\mathcal{S}})_j, K(1_{\mathcal{T}})_j) + \otimes_{\lambda_j}(s, t) \right) \alpha^* \\ &= \alpha \left(\left(K^2 \lambda_j(1, 1) \otimes 1_{\mathcal{S}} \otimes 1_{\mathcal{T}} \right) + \otimes_{\lambda_j}(s, t) \right) \alpha^* \\ &= \alpha \left(\left(K^2 I_{\tau(j)} \otimes 1_{\mathcal{S}} \otimes 1_{\mathcal{T}} \right) + \otimes_{\lambda_j}(s, t) \right) \alpha^* \\ &= \alpha \left(K^2 (1_{\mathcal{S}} \otimes 1_{\mathcal{T}})_{\tau(j)} + \otimes_{\lambda_j}(s, t) \right) \alpha^* \\ &= K^2 \alpha (1_{\mathcal{S}} \otimes 1_{\mathcal{T}})_{\tau(j)} \alpha^* + \alpha \left(\otimes_{\lambda_j}(s, t) \right) \alpha^* \\ &= (K^2 \alpha \alpha^*) 1_{\mathcal{S}} \otimes 1_{\mathcal{T}} + \alpha \left(\otimes_{\lambda_j}(s, t) \right) \alpha^* \end{aligned}$$

which proves that $1_{\mathcal{S}} \otimes 1_{\mathcal{T}}$ is an order unit. Similarly one can prove that $1_{\mathcal{S}} \otimes 1_{\mathcal{T}}$ is in fact matrix order unit. \square

Definition 4.2. Let $\lambda = (\lambda_n)_{n \in \mathbb{N}}$ fulfills conditions (O1)-(O3), and let

$$\mathcal{C}_n^\lambda := \{P \in M_n(\mathcal{S} \otimes \mathcal{T}) : r(1_{\mathcal{S}} \otimes 1_{\mathcal{T}})_n + P \in \mathcal{C}_n, \forall r > 0\}$$

be the Archimedeanization ([16]) of the matrix ordering \mathcal{C}_n for all $n \geq 1$. We call the operator system $(\mathcal{S} \otimes \mathcal{T}, \{\mathcal{C}_n^\lambda\}_{n=1}^\infty, 1_{\mathcal{S}} \otimes 1_{\mathcal{T}})$ the λ -operator system tensor product of \mathcal{S} and \mathcal{T} and denote it by $\mathcal{S} \otimes_\lambda \mathcal{T}$.

Theorem 4.3. The mapping $\lambda : \mathcal{O} \times \mathcal{O} \rightarrow \mathcal{O}$ sending $(\mathcal{S}, \mathcal{T})$ to $\mathcal{S} \otimes_\lambda \mathcal{T}$ is an operator system tensor product in the sense of [13].

Proof. Observe that

- (T1) By definition $(\mathcal{S} \otimes \mathcal{T}, \{\mathcal{C}_n^\lambda(\mathcal{S} \otimes \mathcal{T})\}_{n=1}^\infty, 1_{\mathcal{S}} \otimes 1_{\mathcal{T}})$ is an operator system.
- (T2) For $P \in M_k(\mathcal{S})^+$ and $Q \in M_l(\mathcal{T})^+$, since

$$P \otimes Q = \alpha \otimes_{\lambda_{k+l}} (I_{k+l} \otimes P, I_{k+l} \otimes Q) \alpha^* \in \mathcal{C}_{kl},$$

where $\alpha = (I_{k+l}, 0, \dots, 0) \in M_{kl, \tau(k+l)}$, we have property (T2).

- (T3) For unital completely positive maps $\phi \in \mathcal{S} \rightarrow M_n$ and $\psi \in \mathcal{T} \rightarrow M_m$, using Lemma 3.1(iii) we have $(\phi \otimes \psi)_n(\mathcal{C}_n) \subseteq M_n^+$, thus (T3) follows.
- (T4) Let $\phi \in UCP(\mathcal{S}_1, \mathcal{S}_2)$ and $\psi \in UCP(\mathcal{T}_1, \mathcal{T}_2)$. Then for any element $A \otimes_{\lambda_j}(P, Q) A^* \in \mathcal{C}_n$, where $A \in M_{n, \tau(j)}$, $P \in M_j(\mathcal{S}_1)^+$ and $Q \in M_j(\mathcal{T}_1)^+$, we have

$$\begin{aligned} (\phi \otimes \psi)_n(A \otimes_{\lambda_j}(P, Q) A^*) &= A((\phi \otimes \psi)_{\tau(j)} \otimes_{\lambda_j}(P, Q)) A^* \\ &= A \otimes_{\lambda_j}(\phi_j(P), \psi_j(Q)) A^* \in M_n(\mathcal{S}_2 \otimes_\lambda \mathcal{T}_2)^+. \end{aligned}$$

Thus $(\phi \otimes \psi)_n(\mathcal{C}_n(\mathcal{S}_1, \mathcal{T}_1)) \subseteq \mathcal{C}_n(\mathcal{S}_2, \mathcal{T}_2)$, and hence using [13, Lemma 2.5], we have $\phi \otimes \psi \in UCP(\mathcal{S}_2, \mathcal{T}_2)$. \square

Remark 4.4. If λ is either Kronecker or Schur Product, the cone \mathcal{C}_n^λ coincides with $\mathcal{C}_n^{\max} = \mathcal{C}_n^s$ ([13, 15]).

5. λ -TENSOR PRODUCT OF C^* -ALGEBRAS

We now move on to the algebraic structures for the λ -theory. For this we make use of the condition (W2) (Section 2.1).

An associative algebra A over \mathbb{C} is said to be a completely contractive Banach algebra if it is a complete operator space for which the multiplication map $m_A : A \times A \rightarrow A$ $(a, b) \rightarrow ab$ is jointly completely contractive. That is, $\|[a_{ij}b_{kl}]\| \leq \|[a_{ij}]\| \|[b_{kl}]\|$ for all $[a_{ij}] \in M_n(A)$ and $[b_{kl}] \in M_n(A)$.

Theorem 5.1. *For completely contractive Banach algebras $A_1, A_2, \dots, A_m, \otimes^\lambda A_i$ is a Banach algebra if λ satisfies (W2). Further if each A_i is a Banach $*$ -algebra, and λ satisfies (O1) also, then $\otimes^\lambda A_i$ is in fact a Banach $*$ -algebra. Moreover, if each A_i is approximately unital then $\otimes^\lambda A_i$ is also approximately unital.*

Proof. Let $x, y \in \otimes^\lambda A_i$ with

$$x = \alpha \otimes_{\lambda_r} (u^{(1)}, u^{(2)}, \dots, u^{(m)})\beta \text{ and } y = \gamma \otimes_{\lambda_s} (v^{(1)}, v^{(2)}, \dots, v^{(m)})\delta,$$

where for each $t = 1, 2, \dots, m$

$$\alpha \in M_{1,\tau(r)}, \beta \in M_{\tau(r),1}, \gamma \in M_{1,\tau(s)}, \delta \in M_{\tau(s),1}$$

$$u^{(t)} := \sum_{(i_t, j_t)} \varepsilon_{i_t, j_t}^{[r]} \otimes u_{i_t, j_t}^{(t)}, \quad v^{(t)} := \sum_{(k_t, l_t)} \varepsilon_{k_t, l_t}^{[s]} \otimes v_{k_t, l_t}^{(t)}.$$

Then using Property (W2), there exists $S \in M_{\tau(r)\tau(s), \tau(rs)}, T \in M_{\tau(rs), \tau(r)\tau(s)}$ with $\|S\|, \|T\| \leq 1$ such that

$$\begin{aligned} xy &= \left(\sum_{i_m, j_m} \dots \sum_{i_1, j_1} \alpha \lambda_r(\varepsilon_{i_1, j_1}^{[r]}, \dots, \varepsilon_{i_m, j_m}^{[r]}) \beta \otimes u_{i_1, j_1}^{(1)} \otimes \dots \otimes u_{i_m, j_m}^{(m)} \right) \\ &\quad \left(\sum_{k_m, l_m} \dots \sum_{k_1, l_1} \gamma \lambda_s(\varepsilon_{k_1, l_1}^{[s]}, \dots, \varepsilon_{k_m, l_m}^{[s]}) \delta \otimes v_{k_1, l_1}^{(1)} \otimes \dots \otimes v_{k_m, l_m}^{(m)} \right) \\ &= \sum_{i_m, j_m} \dots \sum_{i_1, j_1} \sum_{k_m, l_m} \dots \sum_{k_1, l_1} ((\alpha \otimes \gamma)(\lambda_r(\varepsilon_{i_1, j_1}^{[r]}, \dots, \varepsilon_{i_m, j_m}^{[r]}) \otimes \lambda_s(\varepsilon_{k_1, l_1}^{[s]}, \dots, \varepsilon_{k_m, l_m}^{[s]})) \\ &\quad (\beta \otimes \delta) \otimes u_{i_1, j_1}^{(1)} v_{k_1, l_1}^{(1)} \otimes \dots \otimes u_{i_m, j_m}^{(m)} v_{k_m, l_m}^{(m)}) \\ &\stackrel{(\text{W2})}{=} (\alpha \otimes \gamma) S \left(\sum_{i_m, j_m} \dots \sum_{i_1, j_1} \sum_{k_m, l_m} \dots \sum_{k_1, l_1} (\lambda_{rs}(\varepsilon_{i_1, j_1}^{[r]} \otimes \varepsilon_{k_1, l_1}^{[s]}, \dots, \varepsilon_{i_m, j_m}^{[r]} \otimes \varepsilon_{k_m, l_m}^{[s]}) \right. \\ &\quad \left. \otimes u_{i_1, j_1}^{(1)} v_{k_1, l_1}^{(1)} \otimes \dots \otimes u_{i_m, j_m}^{(m)} v_{k_m, l_m}^{(m)}) T (\beta \otimes \delta) \right) \\ &= (\alpha \otimes \gamma) S \left(\sum_{i_m, j_m} \dots \sum_{i_1, j_1} \sum_{k_m, l_m} \dots \sum_{k_1, l_1} (\lambda_{rs}(\varepsilon_{(i_1-1)s+k_1, (j_1-1)s+l_1}^{[rs]}, \dots, \right. \\ &\quad \left. \varepsilon_{(i_m-1)s+k_m, (j_m-1)s+l_m}^{[rs]}) \otimes u_{i_1, j_1}^{(1)} v_{k_1, l_1}^{(1)} \otimes \dots \otimes u_{i_m, j_m}^{(m)} v_{k_m, l_m}^{(m)}) T (\beta \otimes \delta) \right) \\ &= (\alpha \otimes \gamma) S(\otimes_{\lambda_{rs}}(z^{(1)}, \dots, z^{(m)})) T(\beta \otimes \delta); \end{aligned} \tag{2}$$

where $z^{(t)} = \sum_{i_t, j_t} \sum_{k_t, l_t} \varepsilon_{(i_t-1)s+k_t, (j_t-1)s+l_t}^{[rs]} \otimes u_{i_t, j_t}^{(t)} v_{k_t, l_t}^{(t)} = u^{(t)} \otimes v^{(t)}$; $t = 1, \dots, m$.

Thus,

$$\begin{aligned} \|xy\|_{\lambda,1} &\leq \|\alpha \otimes \gamma\| \|S\| \|z^{(1)}\| \|z^{(2)}\| \dots \|z^{(m)}\| \|T\| \|\beta \otimes \delta\| \\ &\leq \|\alpha\| \|\gamma\| \|v^{(1)}\| \|w^{(1)}\| \dots \|v^{(m)}\| \|w^{(m)}\| \|\beta\| \|\delta\| \end{aligned}$$

making $\otimes^\lambda A_i$, and hence $\otimes^\lambda A_i$ a Banach algebra.

If λ satisfies (O1), $*$ -part follows from Lemma 3.1(i) and definition of $\|\cdot\|_{\lambda,1}$.

One can easily verify that $\|\cdot\|_{\lambda,1} \leq \|\cdot\|_\gamma$, giving that $\|\cdot\|_{\lambda,1}$ is an admissible cross norm on $\otimes_m A_i$. Therefore, $\otimes^\lambda A_i$ has a bounded approximate identity whenever each A_i is approximately unital. \square

In particular, we have the following well known result (see[14, 17]):

Corollary 5.2. $\otimes^\otimes A_i$, the Projective tensor product and $\otimes^\odot A_i$, the Schur tensor product are Banach $*$ -algebras with a bounded approximate identity. However, $\otimes^\bullet A_i$, the Haagerup tensor product is a Banach algebra.

In general, λ -tensor product of operator spaces is not injective. Since $(\otimes^\lambda A_i)^* = CB_\lambda(A_1 \times A_2 \cdots A_m, \mathbb{C})$ [19, Proposition 4.10] completely isometrically, so the proof of [14, Theorem 5] can be adopted in this case to show the injectivity of λ -tensor product for the closed ideals, i.e.

Proposition 5.3. Let I_i be closed two-sided ideals in C^* -algebras A_i for $i = 1, 2, \dots, m$, then $\otimes^\lambda I_i$ is a closed two-sided $*$ -ideal of $\otimes^\lambda A_i$.

Lemma 5.4. Let W_1, W_2, \dots, W_m be completely complemented subspaces of the operator spaces V_1, V_2, \dots, V_m complemented by cb projection having cb norm equal to 1, respectively. Then $\otimes^\lambda W_i$ is a closed subspace of $\otimes^\lambda V_i$.

Proof. Using the assumption, there are cb projections P_1, P_2, \dots, P_m from V_1 onto W_1, V_2 onto W_2, \dots, V_m onto W_m with $\|P_1\|_{cb} = \|P_2\|_{cb} = \dots = \|P_m\|_{cb} = 1$. Therefore, by the functoriality of the λ - tensor product([19, Proposition 6.1], $\otimes_{i=1}^m P_i : \otimes^\lambda V_i \rightarrow \otimes^\lambda W_i$ is a completely bounded map and $\|\otimes_{i=1}^m P_i\|_{cb} \leq 1$. Since, for $u \in \otimes_{i=1}^m V_i$, $\otimes_{i=1}^m P_i(u) = u$, so $\|u\|_{\otimes^\lambda W_i} \leq \|u\|_{\otimes^\lambda V_i}$. Hence $\otimes^\lambda W_i$ is a closed subspace of $\otimes^\lambda V_i$. \square

Since there is a conditional expectation from a C^* -algebra A onto a finite dimensional C^* -subalgebra of A , so by the above Lemma for finite dimensional C^* -algebras, λ -tensor product of operator spaces is injective. In general $\|\cdot\|_\lambda$ need not be injective.

However, for $\widehat{\otimes}$, we have something partial:

Proposition 5.5. Let A_0 and B_0 be closed $*$ -subalgebras of A and B , respectively, then $A_0 \widehat{\otimes} B_0$ is (isomorphic to) closed $*$ -subalgebra of $A \widehat{\otimes} B$.

Proof. Let I denote the closure of $A_0 \otimes B_0$ in $A \widehat{\otimes} B$, so that I is a closed $*$ -subalgebra of $A \widehat{\otimes} B$. We first claim that $\|u\|_{A \widehat{\otimes} B} \leq \|u\|_{A_0 \otimes B_0} \leq 2\|u\|_{A \widehat{\otimes} B}$ for $u \in A_0 \otimes B_0$. Choose $f \in (A_0 \widehat{\otimes} B_0)^*$ such that $f(u) = \|u\|_{A_0 \otimes B_0}$ with $\|f\| = 1$. Let ϕ_0 be the jcb bilinear form on $A_0 \times B_0$ corresponding to f . By ([7, Corollary 3.10]), $\phi_0 : A_0 \times B_0 \rightarrow \mathbb{C}$ extends to a jcb bilinear form $\phi : A \times B \rightarrow \mathbb{C}$ such that $\|\phi\|_{jcb} \leq 2\|\phi_0\|_{jcb}$. Therefore $\|\tilde{f}\| \leq 2$, where \tilde{f} is the linear functional on $A \widehat{\otimes} B$ corresponding to ϕ , and thus the claim. Now consider the identity map $i : (A_0 \otimes B_0, \|\cdot\|_{A_0 \otimes B_0}) \rightarrow (A \otimes B, \|\cdot\|_{A \otimes B})$ which is linear and continuous by the last claim. So it can be extended to $\tilde{i} : A_0 \widehat{\otimes} B_0 \rightarrow A \widehat{\otimes} B$. We now show that $A_0 \widehat{\otimes} B_0$ is isomorphic to I . For the injectivity of \tilde{i} , by [10, Theorem 2], it is enough to show that it is injective on $A_0 \otimes B_0$ but this follows directly by the last inequality. Again, by the last inequality, \tilde{i}^{-1} is continuous. For onto-ness, let $u \in I$. There is

a sequence $u_n \in A_0 \otimes B_0$ converging to u in $\|\cdot\|_{A \widehat{\otimes} B}$ -norm. The sequence $\{u_n\}$ becomes Cauchy with respect to $\|\cdot\|_{A_0 \widehat{\otimes} B_0}$ -norm by the last claim, so it converges, say, to u' . Clearly, $\tilde{i}(u') = u$. Thus $A_0 \widehat{\otimes} B_0$ can be regarded as a closed $*$ -subalgebra of $A \widehat{\otimes} B$. \square

Proposition 5.6. *For C^* -algebras A and B , any λ -cb bilinear form ϕ on $A \times B$ can be extended uniquely to $\tilde{\phi}$ on $A^{**} \times B^{**}$ such that $\|\phi\|_\lambda = \|\tilde{\phi}\|_\lambda$.*

Proof. Since $\phi : A \times B \rightarrow \mathbb{C}$ is λ -cb bilinear form. It is in particular bounded bilinear form and thus determines a unique separately normal bilinear form $\tilde{\phi} : A^{**} \times B^{**} \rightarrow \mathbb{C}$ by [6, Corollary 2.4]. For $k \in \mathbb{N}$, consider the map $\tilde{\phi}_k : M_k(A^{**}) \times M_k(B^{**}) \rightarrow M_{\tau(k)}$ taking $\tilde{\phi}_k(a_1 \otimes m, a_2 \otimes m') = \lambda_k(a_1, a_2) \otimes \tilde{\phi}(m, m')$. Let $a^{**} = [a_{ij}^{**}] \in M_k(A^{**})$ and $b^{**} = [b_{ij}^{**}] \in M_k(B^{**})$ with $\|a^{**}\| \leq 1$ and $\|b^{**}\| \leq 1$. Since the unit ball of $M_k(A)$ is w^* -dense in the unit ball of $M_k(A^{**})$, so we obtain a net $(a_\lambda) = (a_{ij}^\lambda)$ (resp., $(b_\nu) = (b_{ij}^\nu)$) in $M_k(A)$ (resp., $M_k(B)$) which is w^* -convergent to a^{**} (resp., b^{**}) with $\|a_\lambda\| \leq 1$ (resp., $\|b_\nu\| \leq 1$). Therefore, $\widehat{a_{ij}^\lambda}$ is w^* -convergent to a_{ij}^{**} for each i, j . Now by the separate normality of $\tilde{\phi}$, we have $\|\lambda_k \otimes \tilde{\phi}(\sum_{i,j} \epsilon_{ij} \otimes a_{ij}^{**}, \sum_{p,l} \epsilon_{pl} \otimes b_{pl}^{**})\| = \|\sum_{i,j,p,l} \lambda_k(\epsilon_{ij}, \epsilon_{pl}) \otimes \tilde{\phi}(a_{ij}^{**}, b_{pl}^{**})\| = \|\lim_\lambda \lim_\nu \sum_{i,j,p,l} \lambda_k(\epsilon_{ij}, \epsilon_{pl}) \otimes \tilde{\phi}(a_{ij}^\lambda, b_{pl}^\nu)\| = \lim_\lambda \lim_\nu \|\sum_{i,j,p,l} \lambda_k(\epsilon_{ij}, \epsilon_{pl}) \otimes \phi(a_{ij}^\lambda, b_{pl}^\nu)\| \leq \|\lambda_k \otimes \phi\|$ for each $k \in \mathbb{N}$. Thus $\|\tilde{\phi}_k\| \leq \|\phi_k\| \leq \|\phi\|_\lambda$ for every $k \in \mathbb{N}$. Clearly, $\|\phi\|_\lambda \leq \|\tilde{\phi}\|_\lambda$ as ϕ being the restriction of $\tilde{\phi}$. Hence $\|\phi\|_\lambda = \|\tilde{\phi}\|_\lambda$. \square

For a tensor norm α and a closed ideal J of $A \otimes^\alpha B$, we try to find out whether $a \otimes b \in J_{\min}$ implies that $a \otimes b \in J$. This question stems from the study of the elusive nature of the Haagerup tensor product of C^* -algebras, it was resolved for the Haagerup tensor product in ([1, Theorem 4.4]) and for the operator space projective tensor product in [14, Theorem 6]. We present here a unified approach. For C^* -algebras A and B , assume that $\|\cdot\|_\lambda \geq \|\cdot\|_{\min}$ on $A \otimes B$, so there will be a identity map i from $A \otimes^\lambda B$ into $A \otimes^{\min} B$.

Lemma 5.7. *Let M and N be von Neumann algebras and let L be a closed ideal in $M \otimes^\lambda N$, where the sequence $\lambda = (\lambda_n)_{n \in \mathbb{N}}$ satisfies condition (W2). If $1 \otimes 1 \in L_{\min}$ then $1 \otimes 1 \in L$, and L equals $M \otimes^\lambda N$.*

Proof. Since $1 \otimes 1 \in L_{\min}$, for a given $\epsilon = \frac{1}{2}$, there exists $w \in L$ such that $\|i(w) - 1 \otimes 1\|_{\min} < \frac{1}{2}$. Let $w = \sum_{t=1}^{\infty} \alpha_t \otimes_{\lambda_{r_t}} (u^{(t)}, v^{(t)}) \beta_t$; be a norm convergent representation

in $M \otimes^\lambda N$ where for $t = 1, 2, \dots$, $r_t \in \mathbb{N}$,

$$\alpha_t \in M_{1, \tau(r_t)}, \quad \beta_t \in M_{\tau(r_t), 1}, \quad u^{(t)} := \sum_{(i_t, j_t)} \varepsilon_{i_t, j_t}^{[r_t]} \otimes u_{i_t, j_t}^{(t)}, \quad v^{(t)} := \sum_{(k_t, l_t)} \varepsilon_{k_t, l_t}^{[r_t]} \otimes v_{k_t, l_t}^{(t)}.$$

By [11, Theorem 8.3.5], there exist sequences $\{x_{i_t, j_t}^{(t)}\} \in Z(M)$, $\{y_{i_t, j_t}^{(t)}\} \in Z(N)$, $\{\phi_n\} \in P(M)$ and $\{\psi_n\} \in P(N)$ such that

$$(3) \quad \lim_{n \rightarrow \infty} \|\phi_n(u_{i_t, j_t}^{(t)}) - x_{i_t, j_t}^{(t)}\| = \lim_{n \rightarrow \infty} \|\psi_n(v_{k_t, l_t}^{(t)}) - y_{i_t, j_t}^{(t)}\| = 0 \quad (t = 1, 2, \dots)$$

where $P(M)$ denotes the set of all mappings $\phi : M \rightarrow M$ such that, for $m \in M$, $\phi(m)$ is in the convex hull of the set $\{\text{umu}^* : u \in U(M)\}$.

For each $n \in \mathbb{N}$, using the contractive maps $\phi_n \otimes \psi_n$ on $M \otimes^\lambda N$ ([19, Proposition 6.1]) and invariance of ideal L under $\phi_n \otimes \psi_n$, we have for all positive integers $k \leq l$

$$\begin{aligned}
& \left\| \sum_{t=k}^l \alpha_t \left(\sum_{(k_t, l_t)} \sum_{(i_t, j_t)} \lambda_{r_t}(\varepsilon_{i_t, j_t}^{[r_t]}, \varepsilon_{k_t, l_t}^{[r_t]}) \otimes x_{i_t, j_t}^{(t)} \otimes y_{k_t, l_t}^{(t)} \right) \beta_t \right\|_{\lambda, 1} \\
& \leq \left\| \sum_{t=k}^l \alpha_t \left(\sum_{(k_t, l_t)} \sum_{(i_t, j_t)} \lambda_{r_t}(\varepsilon_{i_t, j_t}^{[r_t]}, \varepsilon_{k_t, l_t}^{[r_t]}) \otimes x_{i_t, j_t}^{(t)} \otimes y_{k_t, l_t}^{(t)} \right) \beta_t \right. \\
& \quad \left. - \sum_{t=k}^l \alpha_t \left(\sum_{(k_t, l_t)} \sum_{(i_t, j_t)} \lambda_{r_t}(\varepsilon_{i_t, j_t}^{[r_t]}, \varepsilon_{k_t, l_t}^{[r_t]}) \otimes \phi_n(u_{i_t, j_t}^{(t)}) \otimes \psi_n(v_{k_t, l_t}^{(t)}) \right) \beta_t \right\|_{\lambda, 1} \\
& \quad + \left\| \sum_{t=k}^l \alpha_t \left(\sum_{(k_t, l_t)} \sum_{(i_t, j_t)} \lambda_{r_t}(\varepsilon_{i_t, j_t}^{[r_t]}, \varepsilon_{k_t, l_t}^{[r_t]}) \otimes \phi_n(u_{i_t, j_t}^{(t)}) \otimes \psi_n(v_{k_t, l_t}^{(t)}) \right) \beta_t \right\|_{\lambda, 1} \\
& \leq \left\| \sum_{t=k}^l \alpha_t \left(\sum_{(k_t, l_t)} \sum_{(i_t, j_t)} \lambda_{r_t}(\varepsilon_{i_t, j_t}^{[r_t]}, \varepsilon_{k_t, l_t}^{[r_t]}) \otimes u_{i_t, j_t}^{(t)} \otimes v_{k_t, l_t}^{(t)} \right) \beta_t \right\|_{\lambda, 1} \quad (\text{Using 3}) \\
& \longrightarrow 0 \text{ as } n \rightarrow \infty, \text{ being the partial sum of } w.
\end{aligned}$$

Therefore, one can define an element

$$z = \sum_{t=1}^{\infty} \alpha_t \left(\sum_{(k_t, l_t)} \sum_{(i_t, j_t)} \lambda_{r_t}(\varepsilon_{i_t, j_t}^{[r_t]}, \varepsilon_{k_t, l_t}^{[r_t]}) \otimes x_{i_t, j_t}^{(t)} \otimes y_{k_t, l_t}^{(t)} \right) \beta_t \in Z(M) \otimes^\lambda Z(N).$$

For sufficiently large choice of n and $\epsilon > 0$, we can easily that

$$(4) \quad \|\phi_n \otimes \psi_n(w) - z\|_\lambda < \epsilon.$$

Since L is left invariant by $\phi_n \otimes \psi_n$ for each n , so

$$z = \lim_{n \rightarrow \infty} (\phi_n \otimes \psi_n(w) \in L \cap (Z(M) \otimes^\lambda Z(N)).$$

It is easy to show that $i \circ (\phi_n \otimes^\lambda \psi_n) = (\phi_n \otimes^{\min} \psi_n) \circ i$.

$$\begin{aligned}
& \|(\phi_n \otimes^{\min} \psi_n)(i(w)) - 1 \otimes 1\|_{\min} = \|(\phi_n \otimes^{\min} \psi_n)(i(w)) - (\phi_n \otimes^{\min} \psi_n)(i(1 \otimes 1))\|_{\min} \\
(5) \quad & \leq \|i(w) - 1 \otimes 1\|_{\min} < \frac{1}{2}.
\end{aligned}$$

By (4), we have

$$(6) \quad \|i \circ (\phi_n \otimes^\lambda \psi_n)(w) - i(z)\|_{\min} < \epsilon, \text{ for sufficiently large } n.$$

Then the inequality

$$(7) \quad \|i(z) - 1 \otimes 1\|_{\min} \leq \frac{1}{2} < 1.$$

is a consequence of (5), (6) and triangle inequality, and so $i(z)$ is invertible in $L_{\min} \cap (Z(M) \otimes^{\min} Z(N))$. Using the same trick as opted in ([12, Theorem 2.11.6, Lemma 2.11.1] and the fact that $Z(M)$ is a nuclear C^* -algebra [2, Proposition

[1] we get $Z(M) \otimes^\lambda Z(N)$ is semisimple. The regularity of $Z(M) \otimes^\lambda Z(N)$ follows from [12, Lemma 4.2.19]. Since $i(z)$ is invertible in $L_{\min} \cap (Z(M) \otimes^{\min} Z(N))$, i.e. invertible in both L_{\min} and $Z(M) \otimes^{\min} Z(N)$, so $0 \notin \sigma_{Z(M) \otimes^{\min} Z(N)}(i(z))$. So by [12, Exercise 4.8.12] $0 \notin \sigma_{Z(M) \otimes^\lambda Z(N)}(z)$. Hence z is invertible in $Z(M) \otimes^\lambda Z(N)$, so there exists $w \in Z(M) \otimes^\lambda Z(N)$ such that $zw = wz = 1 \otimes 1$. Since $z \in L$, and L being an ideal, so $1 \otimes 1 \in L$. \square

Theorem 5.8. *Let $\lambda = (\lambda_n)_{n \in \mathbb{N}}$ fulfills (W2), A and B be C^* -algebras and let J be a closed ideal in $A \otimes^\lambda B$. If $a \otimes b \in J_{\min}$ then $a \otimes b \in J$. In particular $A \otimes^\lambda B$ is a $*$ -semi-simple Banach algebra provided $\lambda = (\lambda_n)_{n \in \mathbb{N}}$ further fulfills (O1).*

Proof. Suppose that $a, b \geq 0$ and $a \otimes b \in J_{\min}$ but not in J . So by Hahn Banach theorem there exists $\phi \in (A \otimes^\lambda B)^*$ such that $\phi(J) = 0$ and $\phi(a \otimes b) \neq 0$. But $(A \otimes^\lambda B)^* = CB_\lambda(A \times B, \mathbb{C})$ so $\phi(x \otimes y) = \tilde{\phi}(x, y)$ for some $\tilde{\phi} \in CB_\lambda(A \times B, \mathbb{C})$ and for all $x \in A, y \in B$. By Proposition 5.6, we have $\tilde{\phi}^{**} : A^{**} \times B^{**} \rightarrow \mathbb{C}$ a λ -completely bounded operator satisfying $\|\tilde{\phi}^{**}\|_\lambda = \|\tilde{\phi}\|_\lambda$. Let L be the closed ideal

in $A^{**} \otimes^\lambda B^{**}$ generated by J . Let $u = \sum_{j=1}^{\infty} \alpha_j \otimes_{\lambda_{k_j}} (x_1^j, x_2^j) \beta_j$ be a norm convergent

sum in $A \otimes^\lambda B$ representing a fixed but an arbitrary element of J . Since ϕ annihilates J , so $\sum_{j=1}^{\infty} \alpha_j \phi_{\lambda_{k_j}} (\otimes_{\lambda_{k_j}} (x_1^j, x_2^j)) \beta_j = \sum_{j=1}^{\infty} \sum_{i_1, i_2} \alpha_j \lambda_{k_j} (a_{i_1}, a_{i_2}) \otimes \phi(x_{i_1}^j \otimes x_{i_2}^j) \beta_j = 0$. Let

$u, v \in A$ and $s, t \in B$ then we have $\sum_{j=1}^{\infty} \sum_{i_1, i_2} \alpha_j \lambda_{k_j} (a_{i_1}, a_{i_2}) \otimes \phi(ux_{i_1}^j v \otimes sx_{i_2}^j t) \beta_j = 0$. Let $M = A^{**}$ and $N = B^{**}$ be the von Neumann algebras generated by A

and B . For each $n \in \mathbb{N}$ and $u \in M$, define $w_n(u) = \sum_{j=1}^n \sum_{i_1, i_2} \alpha_j \lambda_{k_j} (a_{i_1}, a_{i_2}) \otimes \phi(ux_{i_1}^j v \otimes sx_{i_2}^j t) \beta_j$. We will claim that $\{w_n\}$ is a Cauchy sequence. To see this,

let $m < n$, $|w_n(u) - w_m(u)| = \left| \sum_{j=m+1}^n \sum_{i_1, i_2} \alpha_j \lambda_{k_j} (a_{i_1}, a_{i_2}) \otimes \phi(ux_{i_1}^j v \otimes sx_{i_2}^j t) \beta_j \right| \leq$

$\|u\| \|s\| \|t\| \|v\| \sum_{j=m+1}^n \alpha_j \otimes_{\lambda_{k_j}} (x_1^j, x_2^j) \beta_j \|_\lambda$, and so $\{w_n\}$ is a Cauchy sequence with

limit $w \in M_*$ given by $w(u) = \sum_{j=1}^{\infty} \sum_{i_1, i_2} \alpha_j \lambda_{k_j} (a_{i_1}, a_{i_2}) \otimes \phi(ux_{i_1}^j v \otimes sx_{i_2}^j t)$. Again as

in [1], we obtain ϕ annihilates L .

Now for $\epsilon > 0$, let $p_\epsilon \in M$ and $q_\epsilon \in N$ be the spectral projections of a and b respectively for the closed interval $[\epsilon, \infty)$. Since there is a conditional expectation from M onto $p_\epsilon M p_\epsilon$, so $p_\epsilon M p_\epsilon \otimes^\lambda q_\epsilon N q_\epsilon$ is a closed subalgebra of $M \otimes^\lambda N$ by Lemma 5.4. Let $L_0 = L \cap (p_\epsilon M p_\epsilon \otimes^\lambda q_\epsilon N q_\epsilon)$, a closed ideal in $p_\epsilon M p_\epsilon \otimes^\lambda q_\epsilon N q_\epsilon$, and so $(L_0)_{\min}$ is a closed ideal in $p_\epsilon M p_\epsilon \otimes^{\min} q_\epsilon N q_\epsilon$. Now as in [1, Theorem 4.4] and [14, Theorem 6], we get $(L_0)_{\min}$ contains $p_\epsilon \otimes q_\epsilon$ and so by the above Lemma 5.7, $p_\epsilon \otimes q_\epsilon \in L_0$. Hence $L_0 = p_\epsilon M p_\epsilon \otimes^\lambda q_\epsilon N q_\epsilon$, which further implies that $p_\epsilon a \otimes q_\epsilon b \in L$, and so $\phi(p_\epsilon a \otimes q_\epsilon b) = 0$. Letting $\epsilon \rightarrow 0$, we have $\phi(a \otimes b) = 0$, contrary to the choice of ϕ .

In the case when both a and b are arbitrary elements, then one may apply the similar technique as given in [1] to obtain the result.

□

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